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Abstract

Full Text

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On a Method for Investigating Linear Systems of Differential Equations with Quasiperiodic Coefficients

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Let us consider the system of equations

$$\frac{dx_s}{dt} = \sum_{k=1}^{n} P_{sk} x_k \qquad (s = 1, \dots, n), \tag{1}$$

where $P_{sk} = P_{sk}(t)$ are quasiperiodic functions with common frequency basis $\beta_1, \dots, \beta_m$. Thus $P_{sk}(t) = F_{sk}(t, \dots, t)$, where $F_{sk}(u_1, \dots, u_m)$ are continuous periodic functions in the variables u_k with periods $\omega_k = 2\pi/\beta_k$ (1). Form the system

$$\sum_{k=1}^{m} \frac{\partial x_s}{\partial u_k} = \sum_{i=1}^{n} F_{si}(u_1, \dots, u_m) x_i = A_s \qquad (s = 1, \dots, n). \tag{2}$$

Since system (2) is equivalent to the equation

$$\sum_{k=1}^{m} \frac{\partial z}{\partial u_k} + \sum_{s=1}^{n} A_s \frac{\partial z}{\partial x_s} = 0,$$

its solutions exist if the first integrals of the system exist:

$$du_1 = \dots = du_m = \frac{dx_1}{A_1} = \dots = \frac{dx_n}{A_n}.$$

If $x_s(u_1, \dots, u_m)$ is some solution of system (2), then on the diagonal $u_1 = \dots = u_m = t$ it gives a solution $x_s(t, \dots, t)$ of system (1). Conversely, if $x_s(t)$ is an arbitrary solution of system (1), then through it there passes, moreover, an infinite set of solutions of system (2). Clearly, one can always choose such solutions of the system of equations (2) that on the diagonal they give a fundamental system of solutions of the equations (1).

Using the periodicity of the coefficients F_{sk} of system (2), we shall establish in one case the analytic form of the solutions of the system of equations (1).

Let

$$x_{1k}(u_1, \dots, u_m), \dots, x_{nk}(u_1, \dots, u_m) \qquad (k = 1, \dots, n) \tag{3}$$

be n solutions of the system of equations (2) such that the diagonal functions

$$x_{1k}(t, \dots, t), \dots, x_{nk}(t, \dots, t) \qquad (k = 1, \dots, n)$$

constitute a fundamental system of solutions of the equations (1). Then every solution of the system of equations (1) has the form

$$x_s(t, \dots, t) = c_1 x_{s1}(t, \dots, t) + \dots + c_{nx_{sn}}(t, \dots, t) \qquad (s = 1, \dots, n), \tag{4}$$

where c_k are constants.

If, in the functions (3) of some solution of the system of equations (2), we replace all u_j by $u_j + \omega_j$, then, by virtue of the periodicity of the coefficients F_{sk}, we again obtain a solution of the system (2).

The diagonal functions $x_{sk}(t + \omega_1, \dots, t + \omega_m)$ of the solution of the system (2) obtained anew will constitute a solution of the system (1). Therefore, on the basis of (4) we have

$$x_{sk}(t + \omega_1, \dots, t + \omega_m) = a_{1k} x_{s1}(t, \dots, t) + \dots + a_{nk} x_{sn}(t, \dots, t)$$

$$(s, k = 1, \dots, n),$$

where a_{sk} are certain constants.

Let us note that if (3) are n solutions of the system (2), then the functions

$$x_s(u_1, \dots, u_m) = A_1 x_{s1}(u_1, \dots, u_m) + \dots + A_{nx_{sn}}(u_1, \dots, u_m) \tag{5}$$

$$(s = 1, \dots, n)$$

are also a solution of the system (2). Here $A_k = A_k(u_2 - u_1, \dots, u_m - u_1)$ are arbitrary differentiable functions of their arguments.

If in (5) all arguments are increased by the periods ω_k, then in general the functions A_k change.

In what follows we shall assume that the system (1) is such that from relation (4) there follows the relation

$$x_s(t + \omega_1, \dots, t + \omega_m) =$$

$$= c_1 x_{s1}(t+\omega_1, \dots, t+\omega_m) + \dots + c_{n x_{sn}}(t+\omega_1, \dots, t+\omega_m)$$

with the very same constants c_k. We shall call this condition condition (a).

It is easy to give examples of linear systems with quasiperiodic coefficients that satisfy condition (a).

We shall seek a particular solution y_k of the system (1), satisfying condition (a), which satisfies the relation

$$y_k(t+\omega_1, \dots, t+\omega_m) = \lambda y_k(t, \dots, t),$$

where $\lambda \neq 0$ is some constant. We obtain that λ must be a root of the equation

$$|a - \lambda E| = 0, \tag{6}$$

where a is the matrix composed of the constants a_{sk}, and E is the identity matrix. Equation (6) will be called the characteristic equation of the system (1).

Let us note that if the system of solutions (3) of the equations (2) is such that a fundamental system of solutions of the system of equations (1) is determined by the initial conditions $x_{sk}(0, \dots, 0) = 1$, if $s = k$, and $x_{sk}(0, \dots, 0) = 0$, if $s \neq k$, then equation (6) can be written in the form $|X(\omega_1, \dots, \omega_m) - \lambda E| = 0$, where $X(u_1, \dots, u_m)$ is the matrix composed of the solutions (3) of the system (2).

By the usual methods the following propositions are easily proved:

I. The characteristic equation (6) does not change if the system (1) is subjected to a nonsingular linear transformation with quasiperiodic coefficients with a common frequency basis $\beta_1, \dots, \beta_m$.

II. The roots λ_k of the characteristic equation (6) do not depend on the choice of a fundamental system of solutions of the system of equations (1).

III. To each root λ_k of the characteristic equation (6) there corresponds a solution of the equations (1) of the form

$$x_s(t) = \Phi_s(t) e^{\alpha_k t} \qquad (s = 1, \dots, n),$$

where $\Phi_s(t)$ are quasi-periodic functions and

$$\alpha_k = \frac{\delta_1 + \dots + \delta_m}{\omega_1 \delta_1 + \dots + \omega_m \delta_m} \ln \lambda_k;$$

δ_k are certain constants, among which some may be equal to zero.

Thus, in the case of distinct roots of equation (6), the general solution of system (1) has the form

$$x_s(t) = c_1 e^{\alpha_1 t}\Phi_{s1}(t) + \ldots + c_n e^{\alpha_n t}\Phi_{sn}(t) \qquad (s = 1, \ldots, n),$$

where c_k are arbitrary constants.

Let us consider the case of multiple roots of equation (6). Suppose the matrix a has elementary divisors

$$(\lambda - \lambda_1)^{q_1}, \ldots, (\lambda - \lambda_p)^{q_p},$$

where $q_1 + \ldots + q_p = n$. The quantities $\lambda_1, \ldots, \lambda_p$ need not be distinct. In this case we obtain p groups of solutions.

For example, the group corresponding to the root λ_1 has the form

$$x_{k1}(t) = e^{\alpha_1 t}\Phi_{k1}(t),$$

$$x_{k2}(t) = e^{\alpha_1 t}\left[\Phi_{k2}(t) + \frac{r_1(t)}{\lambda_1}\Phi_{k1}(t)\right],$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{kq_1}(t) = e^{\alpha_1 t}\left[\Phi_{kq_1}(t) + \frac{r_1(t)}{\lambda_1}\Phi_{kq_1-1}(t) + \ldots + \frac{r_{q_1-1}(t)}{\lambda_1^{q_1-1}}\Phi_{k1}(t)\right]$$

$$(k = 1, \ldots, n),$$

where

$$r_1(t) = g(t, \ldots, t), \qquad g(u_1, \ldots, u_m) = \frac{\delta_1 u_1 + \ldots + \delta_m u_m}{\omega_1\delta_1 + \ldots + \delta_m\omega_m},$$

$$r_k(t) = \frac{g(g-1)\ldots(g-k+1)}{k!}.$$

Thus, Floquet' s results [2] extend completely to the case of quasi-periodic coefficients.

Let us note that if system (1) satisfies condition (a), then the adjoint system also satisfies this condition.

Hence, in particular, it follows:

IV. *The system of equations (1), by means of a nonsingular linear substitution with quasi-periodic coefficients, can be reduced to a system with constant coefficients.*

Thus, a system of equations (1) with quasi-periodic coefficients satisfying condition (a) behaves in the same way as a system of equations with periodic coefficients.

The indicated method of investigation can also be applied to linear systems with almost-periodic coefficients with integral frequency bases.

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REFERENCES

[1] B. M. Levitan, *Almost-periodic functions*, 1953. [2] M. G. Floquet, *Ann. sci. de l' École norm. sup.*, **12** (1883).

Note: Figure translations are in progress. See original paper for figures.