

# ON ONE CLASS OF SELF-SIMILAR MOTIONS OF AN ULTRARELATIVISTIC GAS

1.\*\* The relativistic equations of motion of an ideal continuous medium have the form

1961

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.75167>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**HYDROMECHANICS**

**V. A. SKRIPKIN**

**ON ONE CLASS OF SELF-SIMILAR MOTIONS OF AN ULTRARELATIVISTIC GAS**

*(Presented by Academician L. I. Sedov on 21 IX 1960)*

1. The relativistic equations of motion of an ideal continuous medium have the form

$$\nabla [(\varepsilon + p)u^i u^k - g^{ik}p] = 0, \quad (1,1)$$

$$\nabla_k(\rho u^k) = 0, \quad (1,2)$$

where the symbol  $\nabla$  denotes the covariant derivative with respect to the coordinate  $x^k$  for the line element  $ds^2 = g_{ik} dx^i dx^k$ ,  $u^i = dx^i/ds$ ;  $p$  is the pressure at rest;  $\varepsilon$  is the proper density of internal energy;  $\rho$  is the residual proper density of matter.

We shall apply dimensional theory <sup>(1,2)</sup> to the study of centrally symmetric motion of an ultrarelativistic gas with the equation of state

$$\varepsilon = 3p. \quad (1,3)$$

Equation (1,3) is an approximate form of the dependence  $\varepsilon = \varepsilon(p, \rho)$  for an ideal monatomic gas <sup>(3)</sup>, when the temperature  $\theta = p/\rho c^2 \gg 1$ . The expansion of  $\varepsilon$  in a series has in this case the form

$$\varepsilon = p \left[ 3 + \frac{1}{2\theta^2} + O(\theta^{-4}) \right]. \quad (1,4)$$

For a photon gas equation (1,3) is satisfied exactly.

The system (1,1) in spherical coordinates for the case of centrally symmetric motion of the medium reduces to two equations

$$\frac{\partial}{\partial t} \left( \frac{\varepsilon + pv^2/c^2}{1 - v^2/c^2} \right) + \frac{\partial}{\partial r} \left( \frac{(p + \varepsilon)v}{1 - v^2/c^2} \right) + \frac{2}{r} \frac{(p + \varepsilon)v}{1 - v^2/c^2} = 0, \quad (1,5)$$

$$\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{(p + \varepsilon)v}{(1 - v^2/c^2)c} \right) + \frac{\partial}{\partial r} \left( \frac{\varepsilon v^2/c^2 + p}{1 - v^2/c^2} \right) + \frac{2\varepsilon v^2/c^2 + p}{r(1 - v^2/c^2)} = \frac{2p}{r}, \quad (1,6)$$

(where  $x^0 = t$  is the time,  $x^1 = r$  is the particle radius), containing three unknown functions  $\varepsilon, p, v = dr/dt$  and one dimensional constant  $c$ , the speed of light in vacuum. Equation (1,3), by eliminating  $\varepsilon$ , makes it possible to solve the system (1,5), (1,6) independently of equation (1,2).

**2.** Let there be, among the constants defining the problem, one constant  $p_0$  with independent dimension of pressure  $[p_0] = ML^{-1}T^{-2}$ . We introduce the dimensionless functions

$$P(\lambda) = \frac{p}{p_0}, \quad V(\lambda) = \frac{v}{c}, \quad \lambda = \frac{r}{ct}. \quad (2,1)$$

Equations (1,5), (1,6), taking (1,3) into account, can be transformed to the form

$$\frac{1}{P} \frac{dP}{d\lambda} = \frac{-\frac{8V^2}{\lambda} - 4\frac{dV}{d\lambda} \frac{2V - \lambda(1 + V^2)}{1 - V^2}}{1 + 3V^2 - 4V\lambda} = \frac{-\frac{8V}{\lambda} + 4\frac{dV}{d\lambda} \frac{2\lambda V - 1 - V^2}{1 - V^2}}{4V - \lambda(3 + V^2)}, \quad (2,2)$$

whence, for  $V(\lambda)$ , we obtain the equation

$$\frac{dV}{d\lambda} = \frac{2V(1 - V^2)(1 - \lambda V)}{\lambda[V^2(\lambda^2 - 3) + 4\lambda V + 1 - 3\lambda^2]}, \quad (2,3)$$

which has the solutions  $V = \pm 1$ ,  $V = 0$ , and  $\lambda = 0$ .

In Fig. 1 the field of integral curves of equation (2,3) is shown qualitatively. The points  $O(0, 0)$ ,  $E(0, 1)$ ,  $B(0, -1)$ ,  $D(0, 1/\sqrt{3})$ ,  $H(1, 1)$  are singular:  $O$ ,  $E$ ,  $B$  are nodes,  $D$  is a saddle, and  $H$  is a singular point of the second order, through which only one integral curve  $V = 1$  passes. Since as  $\lambda \rightarrow \infty$  and  $|V| < 1$ ,  $dV/d\lambda \rightarrow 0$ , it follows that all integral curves passing through the segment  $HK$ , for large  $\lambda$ , run parallel to the horizontal straight lines  $V = \text{const}$ .

Let us turn to the conditions on discontinuity surfaces

$$\frac{(\varepsilon_1 + p_1)(D - v_1)}{c^2 - v_1^2} - \frac{p_1 D}{c^2} = \frac{(\varepsilon_2 + p_2)(D - v_2)}{c^2 - v_2^2} - \frac{p_2 D}{c^2}, \quad (2,4)$$

$$\frac{(\varepsilon_1 + p_1)v_1(D - v_1)}{c^2 - v_1^2} - p_1 = \frac{(\varepsilon_2 + p_2)v_2(D - v_2)}{c^2 - v_2^2} - p_2, \quad (2,5)$$

of which (2,4) is the condition for conservation of the energy flux, and (2,5) of the momentum;  $D$  is the velocity of the discontinuity surface; the indices 1 and 2 refer to quantities on different sides of the jump (4).

For  $\varepsilon = 3p$ , after passing to the dimensionless quantities (2,1), from (2,4), (2,5) we obtain

$$\frac{P_1}{P_2} = \frac{(1 - V_1^2) [(3 + V_2^2)\lambda^* - 4V_2]}{(1 - V_2^2) [(3 + V_1^2)\lambda^* - 4V_1]} = \frac{(1 - V_1^2) [4V_2\lambda^* - 3V_2^2 - 1]}{(1 - V_2^2) [4V_1\lambda^* - 3V_1^2 - 1]}, \quad (2,6)$$

where  $\lambda^* = D/c$ , whence it follows that the quantity

$$L = \frac{(3 + V_2^2)\lambda^* - 4V_2}{4V_2\lambda^* - 3V_2^2 - 1} = \frac{(3 + V_1^2)\lambda^* - 4V_1}{4V_1\lambda^* - 3V_1^2 - 1} \quad (2,7)$$

is conserved in passage through a shock. From (2,7) we find

$$\begin{aligned} & V_2^2(\lambda^* + 3L) - 4V_2(1 + \lambda^*L) + 3\lambda^* + L = \\ & = [V_2(\lambda^* + 3L) + V_1(\lambda^* + 3L) - 4(1 + \lambda^*L)](V_2 - V_1) = 0. \end{aligned}$$

If we discard the root  $V_2 = V_1$ , corresponding to a continuous change of  $V$ , then for  $V_2$ , after substituting the value of  $L$  from (2,7), we obtain

$$V_2 = \frac{3\lambda^{*2} - 1 - 2V_1\lambda^*}{V_1(\lambda^{*2} - 3) + 2\lambda^*}. \quad (2,8)$$

The transformation (2,8) leaves fixed the curves

$$V = \frac{2\lambda^* \pm \sqrt{3}(\lambda^{*2} - 1)}{\lambda^{*2} - 3}, \quad (2,9)$$

which coincide with the isoclines of equation (2,3) on which  $d\lambda/dV = 0$  (the curves  $NH$  and  $ADH$  in Fig. 1). The straight line  $V = 1$  is transformed into the curve  $LH$ , the straight line  $V = -1$  into the curve  $GFH$ , and the straight line  $V = 0$  into the curve  $CDH$ . The separatrix  $BDPE$ , which in a neighborhood of the point  $D$  is represented by a segment of the series

$$V = 3 \left( \lambda - \frac{1}{\sqrt{3}} \right) + \frac{\sqrt{3}}{2} \left( \lambda - \frac{1}{\sqrt{3}} \right)^2 + \dots, \quad (2,10)$$

is mapped onto the curve  $GDR$ . Points of the triangular region  $GOLH$  pass into points for which  $|V| > 1$ , which is devoid of physical meaning. The point

Fig. 1

Figure 1: Fig. 1

$H$  corresponds to the whole segment of the straight line  $HK$ . Let us note that the transformation

(2.8) refers only to the value of the velocity  $V$  and is performed for each point at an unchanged value of  $\lambda^*$ . The point  $O$  ( $\lambda = 0, V = 0$ ) corresponds to the central point  $r = 0$  for  $t > 0$ . The value  $\lambda = \infty$  corresponds to the value  $t = 0$  for any  $r > 0$ ; here  $-1 \leq V(\infty) = V_0 \leq 1$  represents the initial values of the velocity.

Let us now consider specific physical problems that can be solved with the aid of the theory developed.

**Problem of collapse.** Given are  $p(0, r) = p_0, \varepsilon(0, r) = 3p_0, v(0, r) = cV_0 < 0$ , and the speed of the discontinuity surface  $D = c\lambda^*$ . The maximum values of  $\lambda^*$  for which the motion can occur are determined by the points of intersection of the integral curves with the curve  $DR$ . From the region  $CDR$  we arrive, as a result of a jump, in the region  $FDP$ , and from the region  $BDC$  in the region  $GDF$ . In particular, motion with two discontinuities is possible, when we pass from the curve  $DR$  to the separatrix  $PD$ , move along it in the direction from  $P$  to  $D$  and  $B$ , and from the segment  $DB$  pass to points of the curve  $GD$ . To single out a unique solution in this case it is necessary to specify the value  $\lambda_2^*$  also on the second shock wave from the interval  $0 < \lambda_2^* < 1/\sqrt{3}$ . When at the center there is an expanding zone of rest  $v = 0, p = p_1, \varepsilon = 3p_1$ , whose boundary is the shock wave, we pass from the curve  $CD$  to the straight line  $OD$ . Finally, the following solution is also possible: from the straight line  $V = 0$  on the interval  $\infty \geq \lambda \geq 1/\sqrt{3}$  we pass to the separatrix  $DB$ , from which, as a result of a jump, we arrive at the curve  $GD$  and reach the center.

**Fig. 1**

**Problem of expansion.** If at  $t = 0$   $p = p_0, \varepsilon = \varepsilon_0$  and  $v = cV_0 > 0$ , then we can move along the integral curves from  $\lambda = \infty$  to  $\lambda = 1$ . Since the separatrix  $DPE$  is situated entirely to the left of the curve  $CDH$ —the image of the line  $V = 0$ , and in the region  $DHP$   $dV/d\lambda < 0$ , we cannot reach the center along any of the integral curves intersecting  $HM$ , either continuously or by a jump, if we find ourselves in the region  $MDPH$ . Since from any point of the segment  $HM$  we can jump to the point  $H$ , the only possible path to the center  $O$  passes along the line  $V = 1$ , from which it is possible to jump to points of the curve  $LH$  and then continuously to the center, or to the line  $V = \lambda$ , if a vacuum forms at the center. The smallest width of the light wave is determined by the value of  $\lambda^*$  at the point of intersection of the separatrix  $DPE$  and the curve  $LH$ .

piston problem. At  $t = 0$  let  $v = 0, p = p_0, \varepsilon = \varepsilon_0$ , and let the piston velocity be  $c\lambda_0 = cV_0$ . The solution is described by an integral curve going upward from the

point  $(\lambda_0, V_0 = \lambda_0)$  to the curve  $LH$ , then by a sound wave (a discontinuity at  $EH$ ), and, finally, from the point  $H$  by a discontinuity to the straight line  $V = 0$ . If the moving gas consists of photons, then the transition by a discontinuity to the line  $V = 1$  and from this line can be interpreted as a transition from chaotic motion of light quanta to ordered motion and back. If the gas consists of material particles, for which the speed of light is unattainable, then these transitions are associated with the annihilation and creation of matter. For adiabatic motion, from the integral of adiabaticity (formula (2,7'),  $\gamma = 4/3$ ,  $k + s + 3 = 0$  (2)) it follows that

$$R = \text{const} \cdot P^{3/4} \quad \theta = \text{const} \cdot P^{1/4},$$

where  $R = \frac{c^2}{P_0} \rho$  is the number of light quanta per unit proper volume for a photon gas, or the dimensionless density of matter for a gas of material particles.

Mathematical Institute named after V. A. Steklov  
Academy of Sciences of the USSR

Received  
19 IX 1960

## REFERENCES

- <sup>1</sup> L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Moscow, 1957.
- <sup>2</sup> V. A. Skripkin, DAN, **127**, No. 2 (1959).
- <sup>3</sup> W. Pauli, *Theory of Relativity*, Moscow-Leningrad, 1947, § 40.
- <sup>4</sup> V. A. Skripkin, *Astron. Zhurn. AN SSSR*, **37**, issue 2 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*