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Abstract

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ON A GENERALIZATION OF LINDEMANN' S THEOREM

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In 1873 Ch. Hermite [1] proved the transcendence of the number e, and in 1882 F. Lindemann [2] proved the following general theorem on the arithmetic nature of the values of the exponential function at algebraic points:

If $\alpha_1, \dots, \alpha_m$ are any pairwise distinct algebraic numbers, and $a_1, \dots, a_m$ are algebraic numbers, at least one of which is different from zero, then

$$a_1 e^{\alpha_1} + \dots + a_m e^{\alpha_m} \neq 0.$$

It follows from this theorem that the function e^z, at any nonzero algebraic point, assumes a transcendental value, and also that the number π is transcendental.

Lindemann' s theorem is equivalent to the following assertion:

If $\alpha_1, \dots, \alpha_m$ are any algebraic numbers linearly independent over the field of rational numbers, then the numbers $e^{\alpha_1}, \dots, e^{\alpha_m}$ are algebraically independent (i.e., are not connected by any algebraic equation with algebraic coefficients) [4].

In 1929-1930 K. Siegel [3] published a general method, developing and generalizing the ideas of Hermite-Lindemann, which makes it possible to establish transcendence and algebraic independence of values, at algebraic points, of a class of entire functions which he called E-functions, and applied it to solutions of homogeneous linear differential equations of the second order with polynomial coefficients. In particular, he extended Lindemann' s results to Bessel functions.

An entire function

$$f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$$

is called an E-function if: 1) all coefficients c_n belong to an algebraic field K of finite degree over the field of rational numbers; 2) the moduli of the coefficients c_n and of all their conjugates in the field K grow more slowly than any arbitrarily small positive power of n^n; 3) there exists a sequence of natural numbers $\{q_n\}$, growing more slowly than any arbitrarily small positive power of n^n, such that the numbers $q_n c_k$, $k = 0, 1, \dots, n$, are algebraic integers.

In 1954 the author [5,6], by developing Siegel' s method, proved a theorem which enabled him to establish transcendence and algebraic independence of values of E-functions at algebraic points, satisfying linear differential equations with polynomial coefficients of arbitrary orders:

Main theorem. Let the collection of E-functions $f_1(z), \dots, \dots, f_m(z)$ be a solution of a system of m linear differential equations of first order

$$y'_k = Q_{k,0} + \sum_{i=1}^{m} Q_{k,i} y_i, \qquad k = 1, \dots, m,$$

coefficients of which $Q_{k,i}$, $k = 1, \dots, m$; $i = 0, 1, \dots, m$, are rational functions of z, and α is any algebraic number distinct from zero and from the poles of all the functions $Q_{k,i}$. Then, in order that the m numbers $f_1(\alpha), \dots, f_m(\alpha)$ be algebraically independent, it is necessary and sufficient that the functions $f_1(z), \dots, f_m(z)$ be algebraically independent over the field of rational functions ([5],[6]).

For $m = 1$, the main theorem implies:

Theorem 1. Let the E-function $f(z)$ be transcendental and be a solution of the linear differential equation of first order

$$P(z)y' + Q(z)y = R(z), \tag{1}$$

where $P(z)$, $Q(z)$, and $R(z)$ are polynomials, and let α be any algebraic number distinct from zero and from the zeros of the polynomial $P(z)$. Then the number $f(\alpha)$ is transcendental.

This theorem shows that Lindemann' s result on the transcendence of the values of the exponential function remains valid for any transcendental E-function that is a solution of a linear differential equation with polynomial coefficients.

Lemma 1. If $\alpha_1, \dots, \alpha_m$ are arbitrary complex numbers linearly independent over the field of rational numbers, then the functions $e^{\alpha_1 z}, \dots, e^{\alpha_m z}$ are algebraically independent over the field of rational functions.

The proof of the lemma is very simple and well known.

From Lemma 1 and the main theorem follows Lindemann' s theorem.

In the present paper it will be shown that Lindemann's theorem remains valid for any transcendental E-function that is a solution of a linear differential equation of first order with polynomial coefficients, and indeed in a strengthened form.

Theorem 2. Let the E-function $f(z)$ be transcendental and be a solution of the linear differential equation of first order (1), where $P(z)$, $Q(z)$, and $R(z)$ are polynomials, and let $\alpha_1, \dots, \alpha_m$ be arbitrary algebraic numbers, linearly independent over the field of rational numbers, distinct from the zeros of the polynomial $P(z)$. Then the numbers $f(\alpha_1), \dots, f(\alpha_m)$ are algebraically independent. If, however, none of the solutions of the differential equation (1) is a rational function, then the numbers $f(\alpha_1), \ldots, f(\alpha_m)$ are algebraically independent for any distinct algebraic numbers $\alpha_1, \ldots, \alpha_m$, distinct from zero and from the zeros of the polynomial $P(z)$.

Lemma 2. Let an entire transcendental function of order ρ be a solution of equation (1) with polynomial coefficients, and let p and q be the degrees of $P(z)$ and $Q(z)$. Then $\rho \geq 1$ and $\rho = q - p + 1$.

This lemma is a special case of general results on entire-function solutions of linear differential equations with polynomial coefficients, established in Valiron' s monograph ([8]) by the Wiman–Valiron method.

Lemma 3. Let an entire transcendental function $f(z)$ of order ρ not exceeding one be a solution of equation (1) with polynomial coefficients, and suppose that no solution of equation (1) is a rational function, and let $\alpha_1, \ldots, \alpha_m$ be arbitrary distinct nonzero complex numbers. Then the functions

$$y_1 = f(\alpha_1 z), \ldots, y_m = f(\alpha_m z)$$

are algebraically independent over the field of rational functions.

We prove the lemma by induction on m. For $m = 1$ it holds in view of the transcendence of y_1. Assuming it true for $m-1$, we prove that it is true for m. Suppose the contrary. Then there is an equation

$$M \equiv M(y_1, \ldots, y_m) = 0, \tag{2}$$

where M is an irreducible polynomial in m variables with coefficients–polynomials in z, containing y_m. Differentiating equality (2) and then replacing the derivatives $y_1', \ldots, y_m'$ by the right-hand sides of the corresponding

equalities

$$y_k' = -\frac{\alpha_k Q(\alpha_k z)}{P(\alpha_k z)}\, y_k + \frac{\alpha_k R(\alpha_k z)}{P(\alpha_k z)}, \qquad k = 1, \ldots, m, \tag{3}$$

we obtain that M' will also be a polynomial in $y_1, \ldots, y_m$ with coefficients that are rational functions of z. M' is divisible by the irreducible polynomial M, as a polynomial in m variables, since otherwise, eliminating the variable y_m from the two equations $M = 0$ and $M' = 0$, we would obtain an algebraic equation in the field of rational functions among $y_1, \ldots, y_{m-1}$, contrary to the induction hypothesis. But the degrees of M and M' in $y_1, \ldots, y_m$ are equal. Therefore the quotient obtained by dividing M' by M will be a rational function. It follows that the coefficients of equal powers of $y_1, \ldots, y_m$ in M and M' are proportional.

Let us order the terms of M and M' as follows. We regard as leading those terms with greater degree in $y_m, \ldots, y_1$, and in each group of homogeneous terms we arrange them in lexicographic order with respect to these variables. Let

$$A(z) y_n^{k_n} y_{n-1}^{k_{n-1}} \cdots y_1^{k_1}, \qquad 1 \le n \le m, \quad k_n \ge 1, \quad k_i \ge 1, \quad i = 1, \ldots, n-1, \tag{4}$$

where $A(z)$ is a polynomial, be the leading term of M.

We shall show that none of the terms of the form

$$A_j(z) y_n^{k_n - 1} y_{n-1}^{k'_{n-1}} \cdots y_1^{k'_1}, \qquad k'_j = k_j + 1, \quad 1 \le j \le n-1, \tag{5}$$

$$k'_i = k_i, \quad i = 1, \ldots, n-1, \quad i \ne j,$$

can occur in M. Suppose the contrary, and let (5) be one of such terms. In view of (3), the terms of M' of the same order as (4) and (5) will be, respectively,

$$\left[A'(z) - A(z) \sum_{i=1}^{n} \frac{k_i \alpha_i Q(\alpha_i z)}{P(\alpha_i z)} \right] y_n^{k_n} y_{n-1}^{k_{n-1}} \cdots y_1^{k_1}, \tag{6}$$

$$\left[A'_j(z) - A_j(z) \sum_{i=1}^{n} \frac{k'_i \alpha_i Q(\alpha_i z)}{P(\alpha_i z)} \right] y_n^{k_n - 1} y_{n-1}^{k'_{n-1}} \cdots y_1^{k'_1}, \qquad k'_n = k_n - 1. \tag{7}$$

By virtue of the proportionality of the corresponding coefficients in M and M', from (4), (5), (6), and (7) we obtain

$$\frac{A'(z)}{A(z)} - \frac{\alpha_n Q(\alpha_n z)}{P(\alpha_n z)} = \frac{A'_j(z)}{A_j(z)} - \frac{\alpha_j Q(\alpha_j z)}{P(\alpha_j z)}, \qquad j \ne n. \tag{8}$$

Since $\rho < 1$, by Lemma 2, $\rho = 1$ and $p = q$, i.e., the degrees of $P(z)$ and $Q(z)$ are equal. Expanding both sides of equality (8) into partial fractions and comparing the coefficients of z in the zero degree, we obtain $\alpha_j = \alpha_n$, $j \ne n$, which contradicts the conditions of the lemma.

Consider the term of the polynomial M

$$B(z) y_n^{k_n - 1} y_{n-1}^{k_{n-1}} \cdots y_1^{k_1}. \tag{9}$$

After differentiating M, in view of (3), terms of the same order as the term (9) can arise only from the terms (4), (5), and (9). But all terms of the form (5) are equal to zero. Therefore the term of the same order as (9) will be

$$\frac{k_n \alpha_n R(\alpha_n z)}{P(\alpha_n z)} A(z) + B'(z) - B(z) \left[(k_n - 1) \frac{\alpha_n Q(\alpha_n z)}{P(\alpha_n z)} + \sum_{i=1}^{n-1} \frac{k_i \alpha_i Q(\alpha_i z)}{P(\alpha_i z)} \right]. \tag{10}$$

By virtue of the proportionality of the corresponding coefficients M and M, from (4), (6), (9), and (10) we find:

$$\frac{A'(z)}{A(z)} - \frac{\alpha_n Q(\alpha_n z)}{P(\alpha_n z)} = \frac{k_n \alpha_n R(\alpha_n z)}{P(\alpha_n z)} - \frac{A(z)}{B(z)} + \frac{B'(z)}{B(z)},$$

whence

$$P(\alpha_n z)\frac{d}{dz}\frac{B(z)}{A(z)} + \alpha_n Q(\alpha_n z)\frac{B(z)}{A(z)} = k_n \alpha_n R(\alpha_n z). \tag{11}$$

It follows from equality (11) that the rational function

$$\varphi(z) = \frac{1}{k_n}\frac{B(z/\alpha_n)}{A(z/\alpha_n)}$$

is a solution of equation (1). But this contradicts the conditions of the lemma. Hence the lemma is true for m, and, by induction, for all values of m.

We shall now prove Theorem 2. Two cases are possible:

1) Equation (1) has as its solution a rational function $B(z)$ (in particular, for $R(z) \equiv 0$ we have $B(z) \equiv 0$). It follows from the definition that every E-function is an entire function of order not exceeding one. Hence, by Lemma 2, the degrees of $P(z)$ and $Q(z)$ are equal. Therefore, from equation (1) we find

$$f(z) = c\exp\left[-\int \frac{Q(z)}{P(z)}\,dz\right] + B(z) = A(z)\exp az + B(z), \tag{12}$$

where c and a are nonzero constants, and $A(z)$ is a rational function whose zeros and poles are zeros of $P(z)$.

If $\alpha_1, \dots, \alpha_m$ are any algebraic numbers linearly independent over the field of rational numbers and distinct from the zeros of the polynomial $P(z)$, then, by Lemma 1, the functions $e^{a\alpha_1 z}, \dots, e^{a\alpha_m z}$ are algebraically independent over the field of rational functions. Therefore it follows from (12) that $f(\alpha_1 z), \dots, f(\alpha_m z)$ are algebraically independent over the field of rational functions. But these E-functions constitute a solution of the system of differential equations (3). Therefore, by the fundamental theorem (for $z = 1$), the numbers $f(\alpha_1), \dots, f(\alpha_m)$ are algebraically independent.

2) None of the solutions of equation (1) is a rational function. Since $f(z)$, as an E-function, is an entire function of order not exceeding one. Hence, if $\alpha_1, \dots, \alpha_m$ are any distinct algebraic numbers different from zero and from the zeros of $P(z)$, then, by Lemma 3, the functions $f(\alpha_1 z), \dots, f(\alpha_m z)$ are

algebraically independent over the field of rational functions. Then, as in the first case, the numbers $f(\alpha_1), \dots, f(\alpha_m)$ are algebraically independent.

The assertion of Theorem 2 is valid, in particular, for the functions

$$\varphi_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\lambda+1)\dots(\lambda+n)}, \qquad \lambda \neq -1, -2, \dots,$$

with rational values of the parameter λ, satisfying the equation

$$zy' + (\lambda - z)y = \lambda.$$

The last result was obtained by the author as early as 1954 ([5],[7]).

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Note: Figure translations are in progress. See original paper for figures.