A. S. Schwarz

1961

SovietRxiv

Abstract

Full Text

A. S. Schwarz

On the Homotopy Theory of Fibered Spaces

(Presented by Academician P. S. Aleksandrov on 2 VI 1961)

In the present note some results are indicated on higher obstructions to the extension of cross-sections. A duality of fibrations is constructed, generalizing the Spanier-Whitehead duality (^{8}). All fibrations are assumed to be locally trivial.

1. Obstructions to the extension of cross-sections. Let $\mathfrak{B}(E, B, F, p)$ be a fibration whose base is a cell complex, and let $\mathfrak{B}_n(E_n, B, F_n, p_n)$ be the natural system (the Postnikov system) of this fibration (^{4}).

Theorem 1. Suppose that the fiber F of the fibration $\mathfrak{B}$ is aspherical in dimensions $< l$, the base B is aspherical in dimensions $< n-l$, the cohomology groups

$$H^j(B, \pi_{j-1}(F))$$

are trivial in dimensions $l+1 < j < n$, and the characteristic class of the fibration

$$\xi(\mathfrak{B}) \in H^{l+1}(B, \pi_l(F))$$

is equal to zero. Then the fibration $\mathfrak{B}_{n-2}$ is equivalent to a direct product.

With the help of this theorem and the relation, indicated by Hermann (^{4}), between the fibrations $\mathfrak{B}_n$ and the obstructions to the extension of cross-sections, one can describe the obstructions to the extension of cross-sections of the fibration $\mathfrak{B}$ from the n-dimensional skeleton to the n-dimensional skeleton. We give only the result obtained in the simplest case.

Theorem 2. Let $\mathfrak{B}(E, B, S^l, p)$ be a fibration whose fiber is the l-dimensional sphere S^l, and whose base is an n-dimensional closed manifold B, aspherical in dimensions $< n-l$. Then the obstruction $z^n(f)$ to the extension of a cross-section f from the $(n-1)$-dimensional skeleton of the base to the n-dimensional skeleton is determined by the formula

$$p^* z^n(f) = p^*(b) \cup a_f - T(a_f).$$

(Here $a_f \in H^l(E, \mathbb{Z})$ is the cohomology class which cuts out on the fiber the fundamental class of the sphere S^l and satisfies the condition $f^*(a_f) = 0$; $b \in H^{n-l}(B, \pi_{n-1}(S^l))$ is a cohomology class which, for $n < 2l$, does not depend on the cross-section f;

$$T : H^l(X, \mathbb{Z}) \to H^n(X, \pi_{n-1}(S^l))$$

is a partial multivalued operation, determined* by the factor

$$k^n \in H^n(F_{n-2}, \pi_{n-1}(S^l))$$

of the natural system of the sphere S^l; under the conditions of the theorem the element $T(a)$ is uniquely determined for every

$$a \in H^l(E, \mathbb{Z}),$$

and, for $n < 2l$, the mapping

$$T : H^l(E, \mathbb{Z}) \to H^n(E, \pi_{n-1}(S^l))$$

is additive.)

* A cohomology class $a \in H^n(N, G)$ of an arbitrary space N, aspherical in dimensions $> n$, defines a partial multivalued operation

$$A : H^l(X, \pi_l(N)) \to H^n(X, Y),$$

where $\pi_l(N)$ is the first nonzero homotopy group of the space N, by means of the following convention: if $x \in H^l(N, \pi_l(N))$ is the fundamental class of the space N, $\xi \in H^l(X, \pi_l(N))$, and $f : X \to N$ is a mapping for which

$$f^*\xi = x,$$

then one sets

$$A(x) = f^*(a).$$

With the help of Theorem 2 one can obtain the following assertion:

Theorem 3. *If an n-dimensional closed manifold M, aspherical in dimensions $\leq k$, is smoothly embedded in an $(2n-k+1)$-dimensional closed manifold N, for which $H^i(N, \pi_{i-1}(S^{n-k})) = 0$ for $n-k+1 < i \leq n-1$, in such a way that the cycle determined by the manifold M is homologous to zero in N, then on the manifold M there exists a transverse vector field (i.e., there exists a section of the spherical bundle of the normal bundle (E, M, S^{n-k}, p)).*

The proof is based on the fact that the space of the normal bundle E is naturally embedded in the space $N \setminus M$, in which the operation T is trivial for dimensional reasons. Applying Theorem 3 to the case when N is the sphere S^{2n-k+1}, with the aid of the results of Hirsch (^{5}), we conclude that every smooth embedding of an n-dimensional manifold, aspherical in dimensions $\leq k$, in S^{2n-k+1} is regularly homotopic in S^{2n-k+1} to an immersion in S^{2n-k}. (We note that Haefliger (^{6}) proved that a manifold aspherical in dimensions $\leq k$ can be smoothly embedded in S^{2n-k} and immersed in S^{2n-k-1}.)

2. Operations on fibered spaces. The sum of $\mathfrak{B}_1(E_1, B, F_1, p_1)$ and $\mathfrak{B}_2(E_2, B, F_2, p_2)$ with common base B is the fibration $\mathfrak{B}_1 + \mathfrak{B}_2(E, B, F_1 * F_2, p)$, whose space E is defined as the set of triples (e_1, e_2, t) satisfying the condition $p_1(e_1) = p_2(e_2)$, with the identifications $(e_1, e_2, 0) \sim (e_1, e_2', 0)$, $(e_1, e_2, 1) \sim (e_1', e_2, 1)$ (here $e_i, e_i' \in E_i$, $0 \leq t \leq 1$, and the projection $p : E \to B$ is given by the formula $p(e_1, e_2, t) = p_1(e_1) = p_2(e_2)$).

The sum of any set of fibrations with the same base is defined analogously. The operation of addition of fibrations is commutative and associative.

Some results concerning the sum of fibrations, as well as applications of this concept to the computation of the genus of a fibration, may be found in (^{2}). Here we note only the following assertion.

Theorem 4. *In order that there exist a fiber-preserving map $f : X \to E$ of the fibration $\mathfrak{B}(X, Y, F, \pi)$ into the sum $\mathfrak{B}_1 + \dots + \mathfrak{B}_n(E, B, F_1 * \dots * F_n, p)$ of fibrations $\mathfrak{B}_i(E_i, B, F_i, p_i)$, $i = 1, \dots, n$, covering a map of bases $\varphi : Y \to B$ (i.e., satisfying the condition $\varphi\pi = pf$), it is necessary and sufficient that there exist an open covering $\{Y_1, \dots, Y_n\}$ of the space Y such that, for each i, $1 \leq i \leq n$, one can construct a fiber-preserving map of the fibration $(p^{-1}(Y_i), Y_i, F, \pi)$ into $\mathfrak{B}_i$, covering the map $\varphi : Y_i \to B$ (the space Y is assumed normal).*

Let $\mathfrak{B}_i(E_i, B, F_i, p_i)$ $(i = 1, 2)$ be two fibrations with one and the same base. The space of fiber-preserving maps of the fibration $\mathfrak{B}_1$ into the fibration $\mathfrak{B}_2$ that induce the identity map on the base will be denoted by $M(\mathfrak{B}_1, \mathfrak{B}_2)$, the set of path components of this space (the set of homotopy classes of fiber-preserving maps) by $[\mathfrak{B}_1, \mathfrak{B}_2]_0$, and the group $\pi_i(M(\mathfrak{B}_1, \mathfrak{B}_2))$ by $[\mathfrak{B}_1, \mathfrak{B}_2]_i$ $(i \geq 1)$. (The space of maps $M(\mathfrak{B}_1, \mathfrak{B}_2)$ is endowed with the compact-open topology.)

Suppose there exists a topological group G for which B is a classifying space (i.e., there exists a universal principal fibration $\mathfrak{S}(X, B, G, \pi)$). Then, modifying one construction of Onishchik, we associate with a fibration $\mathfrak{B}(E, B, F, p)$ the G-space (^{1})$\mathfrak{D}(\mathfrak{B})$, defined by the principal fibration $(\mathfrak{D}(\mathfrak{B}), E, G)$, which is induced by the fibration $\mathfrak{S}$ and the map $p : E \to B$. (This construction may be regarded as inverse to the Serre construction used by Borel $((^3), p.209)$.) It is easy to verify that $\mathfrak{D}(\mathfrak{B}_1 + \mathfrak{B}_2) = \mathfrak{D}(\mathfrak{B}_1) * \mathfrak{D}(\mathfrak{B}_2)$; one can establish a one-to-one correspondence between the set $[\mathfrak{B}_1, \mathfrak{B}_2]_0$ and the set $[\mathfrak{D}(\mathfrak{B}_1), \mathfrak{D}(\mathfrak{B}_2)]$ of homotopy classes of admissible maps of $\mathfrak{D}(\mathfrak{B}_1)$ into $\mathfrak{D}(\mathfrak{B}_2)$.

3. $\mathfrak{R}$-category. Fix some fibration $\mathfrak{R}(E, B, F, p)$. Let $\mathfrak{R}_k$ be the sum of k copies of the fibration $\mathfrak{R}$. If $\mathfrak{B}_i(E_i, B, F_i, p_i)$

$(i = 1, 2)$ are two fiber spaces with the same base B, then by an $\mathfrak{R}$-mapping of the fiber space $\mathfrak{B}_1$ into the fiber space $\mathfrak{B}_2$ we shall mean a fiber-preserving mapping of the fiber space $\mathfrak{B}_1 + \mathfrak{R}_k$ into the fiber space $\mathfrak{B}_2 + \mathfrak{R}_k$, inducing the identity mapping on the base. Two fiber spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are called $\mathfrak{R}$-equivalent if, for some $k \geq 0$, the fiber spaces $\mathfrak{B}_1 + \mathfrak{R}_k$ and $\mathfrak{B}_2 + \mathfrak{R}_k$ are homotopy equivalent. The sets $[\mathfrak{B}_1 + \mathfrak{R}_k, \mathfrak{B}_2 + \mathfrak{R}_k]$ form, with respect to naturally defined mappings, a direct spectrum, whose limit we denote by $\{\mathfrak{B}_1, \mathfrak{B}_2\}_i$; for $i \geq 1$ the set $\{\mathfrak{B}_1, \mathfrak{B}_2\}_i$ can be endowed with a group structure. The set $\{\mathfrak{B}_1, \mathfrak{B}_2\}_0$ is naturally called the set of homotopy classes of $\mathfrak{R}$-mappings of $\mathfrak{B}_1$ into $\mathfrak{B}_2$. The category whose objects are fiber spaces with base B, and whose morphisms are $\mathfrak{N}$-mappings, will be called the $\mathfrak{N}$-category. We note that $\mathfrak{N}$-equivalence of sphere bundles in the case when $\mathfrak{N}$ is the direct product was used by Borel and Hirzebruch (^{7}).

4. Duality of fiber spaces. Let, for a fiber space $\mathfrak{N}(E, B, F, p)$ fixed by us, the base be a finite polyhedron and the fiber an r-dimensional polyhedron homotopy equivalent to the r-dimensional sphere S^r. Under these conditions, an $\mathfrak{N}$-mapping

$$\mathfrak{B}_1 + \mathfrak{N}_k \to \mathfrak{B}_2 + \mathfrak{N}_k$$

of the fiber space $\mathfrak{B}_1(E_1, B, F_1, p_1)$ into the fiber space $\mathfrak{B}_2(E_2, B, F_2, p_2)$ gives rise to an S-mapping of the fiber F_1 into the fiber F_2 (since the space $F_i * F * \cdots * F$ is homotopy equivalent to the $k(r+1)$-fold suspension over F_i). We shall now define duality of fiber spaces with base B relative to the fiber space $\mathfrak{N}$; this duality generalizes Spanier–Whitehead duality (^{8}) and reduces to it when the base B consists of a single point.

Definition. Fiber spaces $\mathfrak{B}_1(E_1, B, F_1, p_1)$ and $\mathfrak{B}_2(E_2, B, F_2, p_2)$, whose fibers are finite polyhedra, will be called n-dual and we shall write $\mathfrak{B}_2 = D_n(\mathfrak{B}_1)$, if there exists an $\mathfrak{N}$-mapping of the sum $\mathfrak{B}_1 + \mathfrak{B}_2$ of the fiber spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ into the fiber space $\mathfrak{N}_n$, such that the S-mapping thereby defined from the fiber $F_1 * F_2$ into the space $F * \cdots * F$, homotopy equivalent to the $(nr + n - 1)$-dimensional sphere, induces (^{1}) a nondegenerate scalar product of the reduced homology groups $H_i(F_1, A)$ and $H_{nr+n-i-2}(F_2, A)$ for any field A and any $i \geq 0$. We note that, under the conditions of this definition, the fibers F_1 and F_2 are weakly $(nr + n - 1)$-dual in the sense of Spanier–Whitehead.

The usual properties of Spanier–Whitehead duality can be carried over to dual fiber spaces. We record only the following proposition.

Theorem 5. *If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are fiber spaces with base B; $D_n(\mathfrak{B}_1)$ and $D_n(\mathfrak{B}_2)$ are fiber spaces dual to them, then one can establish a one-to-one correspondence between the sets $[\mathfrak{B}_1, \mathfrak{B}_2]_0$ and $[D_n(\mathfrak{B}_2), D_n(\mathfrak{B}_1)]_0$, and an isomorphism between the groups $[\mathfrak{B}_1, \mathfrak{B}_2]_i$ and $[D_n(\mathfrak{B}_2), D_n(\mathfrak{B}_1)]_i$ for $i \geq 1$.*

The duality of fiber spaces constructed here is closely related to the duality of G-spaces indicated by us earlier (^{1}). Namely, if the fiber spaces $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are n-dual relative to the fiber space $\mathfrak{N}$, then the G-spaces $\mathfrak{D}(\mathfrak{B}_1)$ and $\mathfrak{D}(\mathfrak{B}_2)$ (see no. 2) are n-dual relative to the G-space $\mathfrak{D}(\mathfrak{B})$.

In what follows we shall assume that $\mathfrak{N}(E, B, S^0, p)$ is the direct product of the space B with the zero-dimensional sphere. Then the fiber spaces $\mathfrak{B}_1(E_1, B, S^{k-1}, p_1)$ and $\mathfrak{B}_2(E_2, B, S^{l-1}, p_2)$ are $(k + l)$-dual if the fiber space $\mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{N}_s$, for some s, is homotopy equivalent to the direct product. Since the Whitney sum of sphere bundles, as a fiber space, is the sum of fiber spaces in the sense defined above, it follows that the sphere bundle $\mathfrak{B}_1(E_1, B, S^{k-1}, p_1)$ tangent to a k-dimensional manifold B, lying in an n-dimensional Euclidean space, is n-dual to the normal fiber space $\mathfrak{B}_2(E_2, B, S^{n-k-1}, p_2)$. As is known

(⁹), for any fiber space $\mathfrak{B}$ whose fiber is a sphere, one can define (with the aid of Stiefel squares) the Stiefel characteristic classes $w_i(\mathfrak{B})$; it is easy to verify

to say that these classes coincide for $\mathfrak{R}$-equivalent fiberings. The following generalization of Whitney' s duality theorem holds: if $\mathfrak{B}' = D_n(\mathfrak{B})$, then

$$\sum_{i+j=k} w_i(\mathfrak{B}) \cup w_j(\mathfrak{B}') = 0$$

for all $k > 0$.

Voronezh State
University

Received
31 V 1961

References

1. A. S. Shvarts, DAN, 136, No. 1, 43 (1961).
2. A. S. Shvarts, Tr. Moscow Math. Soc., 10 (1961).
3. A. Borel, Collection. Fibre spaces and their applications, IL, 1958.
4. R. Hermann, Illinois J. Math., 4, No. 1, 9 (1960).
5. M. Hirsh, Trans. Am. Math. Soc., 93, No. 2, 242 (1959).
6. A. Haefliger, Bull. Am. Math. Soc., 67, No. 1, 109 (1961).
7. A. Borel, F. Hirzebruch, Am. J. Math., 82, No. 3, 491 (1960).
8. E. Spanier, J. H. C. Whitehead, Mathematika, 2, 56 (1955).
9. R. Thom, Ann. École Norm., 69, 109 (1952).