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Abstract

Full Text

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On the Theory of Optimal Pursuit

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1°. Statement of the problem*

Let $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ be points of a real n-dimensional phase space $\bar{R}^n$, whose equations of motion have, respectively, the form:

$$\dot{x} = f(x, u) = (f^1(x, u), \dots, f^n(x, u)); \tag{1}$$

$$\dot{y} = g(y, v) = (g^1(y, v), \dots, g^n(y, v)), \tag{2}$$

where $u = u(t) = (u^1(t), \dots, u^r(t))$ is the vector controlling the motion of the point x; $v = v(t) = (v^1(t), \dots, v^s(t))$ is the vector controlling the motion of the point y. The control $u(t)$ is chosen from the class of piecewise-continuous vector functions taking values in a given set Ω^r of an r-dimensional vector space, and the control $v(t)$ from the class of piecewise-continuous vector functions with values in some set Ω^s of an s-dimensional vector space. Such controls will be called admissible. The functions $f(x, u)$, $g(y, v)$ are assumed to depend continuously on the arguments (x, u), (y, v) and to be continuously differentiable with respect to all coordinates of the points x, y, respectively. We shall call the point x the pursuing point, and the point y the pursued point.

Suppose that for any admissible control $v(t)$ and given initial conditions

$$x(0) = x_0, \qquad y(0) = y_0 \tag{3}$$

there exists an admissible control $u(t)$ such that the trajectories $x(t), y(t)$ of equations (1), (2), corresponding to these controls and to the initial values (3), satisfy the condition $x(t_1) = y(t_1)$ for some time $t_1 > 0$. We shall assume that, for the chosen $u(t), v(t)$, the equality $x(t) = y(t)$ is impossible for $0 \leq t \leq t_1$. The quantity $T_{uv} = t_1$ (depending on the chosen controls $u(t), v(t)$) will be called the pursuit time. In what follows we shall assume that the initial conditions (3) are fixed.

If the control $v(t)$ of the pursued point is chosen, then the pursuing point should be controlled in such a way that the corresponding pursuit time T_{uv} assumes the minimal value. Suppose that, for any admissible choice of $v(t)$, this minimum is attained for some $u(t)$. Denote it by

$$T_v = \min_u T_{uv}.$$

The pursued point must choose an admissible control $v(t)$ maximizing the quantity T_v. This maximum, if it exists, will be denoted by

$$T = \max_v \min_u T_{uv}.$$

* The work was carried out in L. S. Pontryagin' s seminar on the theory of oscillations and automatic control.

Our problem consists in choosing controls $u(t), v(t)$ such that, for the corresponding pursuit time T_{uv}, the equality $T_{uv} = T$ holds. Such a pair of controls $u(t), v(t)$ will be called an **optimal pair of controls**, and the corresponding trajectories an **optimal pair of trajectories**.

The theorem proved in the present note gives a complete system of necessary conditions satisfied by every optimal pair of trajectories, under one additional assumption: that equation (1) is linear, nondegenerate, and the set Ω^r is a convex closed polyhedron in r-dimensional space (see ([1,2])). Thus equation (1) has the form

$$\dot{x} = f(x, u) = Ax + Bu, \tag{4}$$

where A is a linear operator acting in R^n; B is a linear operator mapping Ω^r into R^n.

2°. Let $\psi = (\psi_1, \dots, \psi_n)$, $\chi = (\chi_1, \dots, \chi_n)$ be two arbitrary covariant vectors of the space R^n. Introduce a scalar function of six vector arguments ψ, x, u, χ, y, v:

$$H(\psi, x, u, \chi, y, v) = \sum_{\alpha=1}^{n} [\psi_\alpha f^\alpha(x, u) + \chi_\alpha g^\alpha(y, v)] = \psi \cdot f(x, u) + \chi \cdot g(y, v).$$

Theorem. Let $u(t), v(t)$ be an optimal pair of controls, and $x(t), y(t)$ the corresponding optimal pair of trajectories of equations (4), (2), and let T be the pursuit time; then there exist continuous nonzero covariant vector-functions

$$\psi(t) = (\psi_1(t), \dots, \psi_n(t)), \quad \chi(t) = (\chi_1(t), \dots, \chi_n(t)), \quad 0 \le t \le T,$$

such that the functions $\psi(t), x(t)$ satisfy the Hamiltonian system

$$\dot{x}^i = f^i(x, u) = \frac{\partial H}{\partial \psi_i}, \qquad \dot{\psi}_i = -\frac{\partial H}{\partial x^i}, \qquad i = 1, \dots, n,$$

and the functions $\chi(t), y(t)$ satisfy the Hamiltonian system

$$\dot{y}^i = g^i(y, v) = \frac{\partial H}{\partial \chi_i}, \qquad \dot{\chi}_i = -\frac{\partial H}{\partial y^i}, \qquad i = 1, \dots, n.$$

For every t on the interval $0 \le t \le T$, the Hamiltonian function

$$\begin{aligned} H(t) &= H(\psi(t), x(t), u(t), \chi(t), y(t), v(t)) \\ &= \max_{u \in \Omega^r} \min_{v \in \Omega^s} H(\psi, x, u, \chi, y, v), \end{aligned} \tag{5}$$

where $H(t) = \text{const} \ge 0$, and at the moment $t = T$ the equality $\psi(T) = -\chi(T)$ holds.

3°. **Proof.** We shall carry out the proof for the case in which, in the initial data (3), $x(0) = x_0 = 0$, i.e., for the case in which the pursuing object starts from the origin. In addition, we shall suppose that the set Ω^r contains the origin. It is easy to show that these assumptions do not restrict generality.

Denote by M_T the set of points of the phase space R^n that can be reached from the origin in time $\le T$, moving along trajectories of equation (4) by means of admissible controls. Let Σ_T denote the boundary of this set. From the results of (^{1,2}) there follow the properties of the sets M_T, Σ_T listed below. M_T is a compact convex set containing interior points; consequently, M_T is homeomorphic to an n-dimensional ball, Σ_T to an $(n-1)$-dimensional sphere. Σ_T consists of precisely those, and only those, points which can be reached from the origin in time $\ge T$. For any T, the set M_T can be represented as a homeomorphic image of the direct product of the $(n-1)$-dimensional sphere S^{n-1} with the interval $0 \le t \le T$, where this homeomorphism maps the lower base $O \times S^{n-1}$ of the direct product to the origin, and the set $t \times S^{n-1}$ to Σ_t, $0 \le t \le T$.

By M we denote the set-theoretic union of the sets M_t,

$$0 \le t \le \infty, \qquad M = \bigcup_{t=0}^{\infty} M_t.$$

Obviously, M is an open set.

Let $u(t), v(t)$, $0 \le t \le T$, be an optimal pair of controls; $x(t), y(t)$ the corresponding optimal pair of trajectories; and T the pursuit time. Since M is an open set, there exists an increasing sequence of times $t_i \to T$, $i \to \infty$, such that all the points $y(t_i) \in M$. From the properties of the sets Σ_T listed above it follows that $y(t_i) \in \Sigma_{\tau_i}$, where $\tau_i > t_i$ and $\tau_i \to T$ as $i \to \infty$. Through the point $y(t_i)$ draw a supporting hyperplane to M_{τ_i}, and let φ^i be a vector orthogonal to this hyperplane and directed toward the side opposite to M_{τ_i}. Choose from the sequence of vectors φ^i a convergent subsequence, whose limit we denote by ψ. In order not to change the notation, assume that the sequence φ^i itself converges. Since $x(t_i) \in \Sigma_{t_i}$, $y(t_i) \in \Sigma_{\tau_i}$, where $\tau_i > t_i$, and M_{τ_i} contains M_{t_i}

strictly inside itself, we have $\varphi^i \cdot (y(t_i) - x(t_i)) > 0$. Consider the functions $\Phi^i(t) = \varphi^i \cdot (y(t) - x(t))$, $\Phi(t) = \lim_{t\to\infty} \Phi^i(t)$. In view of the obvious properties of the function $\Phi(t)$, we conclude that $\Phi(T) \le 0$, or $\varphi \cdot (f(T) - g(T)) \ge 0$. From this inequality it follows that the point $x(T) = y(T)$ is a boundary point of the convex closure D of the set M_T and of the vector $g(T) - f(T)$.

Let $\tilde{y}(t) = y(t) + \varepsilon\delta y(t) + \varepsilon O(\varepsilon)$ be an arbitrary varied trajectory of equation (2) with initial value (3). Denote the pursuit time of the varied trajectory by $T_i = T - \delta T(\varepsilon_i)$, where $\delta T(\varepsilon_i) \ge 0$ and $\delta T(\varepsilon_i) \to 0$ as $\varepsilon_i \to 0$. Through the point $y(T_i) \in M_{\sigma_i}$ draw a supporting hyperplane to M_{σ_i}, and let ξ^i be a vector orthogonal to this hyperplane and directed toward the side opposite to M_{σ_i}. Denote the limit of a convergent subsequence ξ^i by ξ. The point $\tilde{y}_i(T_i) \in \Sigma_{T_i}$, and, since $\sigma_i \ge T_i$, we have $\xi^i \cdot (\tilde{y}_i(T_i) - y(T_i)) \le 0$. Passing to the limit, we obtain

$$\xi \cdot \delta y(T) \le 0. \tag{6}$$

It follows from the inequality obtained that the point $x(-T) = y(T)$ is a boundary point of the convex closure E of the set $\dot{D}$ and of the reachability cone K for the trajectory $y(t)$ (see (3)). Denote by ψ a vector orthogonal to the supporting hyperplane to the set E at the point $x(T) = y(T)$ and directed toward the side opposite to E. Take ψ as the boundary value of the function $\psi(t)$, $0 \le t \le T$: $\psi(T) = \psi$, and define the boundary value of the function $\chi(t)$, $0 \le t \le T$, by the equality $\chi(T) = -\psi$. These conditions uniquely determine $\psi(t), \chi(t)$, $0 \le t \le T$, from the system $\dot{\psi}_i = -\partial H/\partial x^i$, $\dot{\chi}_i = -\partial H/\partial y^i$, $i = 1, \dots, n$. The functions $\psi(t), \chi(t)$ found in this way satisfy the theorem formulated above.

Indeed, the hyperplane passing through the point $y(T)$ and orthogonal to the vector ψ is supporting to the set M_T. Hence it follows (see (1)) that on the whole interval $0 \le t \le T$ the maximum condition is satisfied:

$$\psi(t) \cdot f(x(t), u(t)) = \max_{u\in\Omega^r} [\psi(t) \cdot f(x(t), u))],$$

and, consequently,

$$H(t) = \max_{u\in\Omega'} H(\psi, (t), x(t), u, \chi(t), y(t), v(t)).$$

The minimum condition in equality (5) follows from the inclusion $K \subset E$ (see (3)). Finally, it is not difficult to show that on the entire interval $0 \le t \le T$ the function $H(t) = \text{const} \ge 0$.

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Note: Figure translations are in progress. See original paper for figures.

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