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Abstract

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COHOMOLOGY OF GROUPS OF UNITS IN DIVISION ALGEBRAS

(Presented by Academician L. S. Pontryagin on 18 XI 1960)

Let $\mathfrak{A}$ be a division algebra of rank $n > 1$ over the field of rational numbers Q, let $J = [\tau_1, \dots, \tau_n]$ be a maximal order in $\mathfrak{A}$, and let G be the group of units in J of norm $+1$. The aim of the present note is to prove the following periodicity property of the cohomology of the group G:

For any G-module A and all sufficiently large k there is an isomorphism

$$H^k(G, A) \approx H^{n-1+k}(G, A) \tag{1}$$

(more precisely, see Theorem 1).

1. The proof of (1) is based on Grothendieck' s theory of G-bundles ([1]). Let X be a topological space with a group of transformations G; let Y be the quotient space X/G, and let $f : X \to Y$ be the canonical map. Denote by $C^{X(G)}$ (respectively, C^Y) the abelian category of G-bundles of abelian groups over X (respectively, bundles of abelian groups over Y). Denote by Γ^G the functor assigning to a G-bundle A the group of its G-invariant sections over X, and by f_*^G the functor from $C^{X(G)}$ to C^Y given by the formula $(f_*^G A)V = (A(f^{-1}(V)))^G$. The right derived functors of Γ^G will be denoted by $H^p(X; G, A)$, and the right derived functors of f_*^G by $\mathcal{H}^p(G, A)$. The functor Γ^G can be decomposed in two ways into a composition of functors $\Gamma^G = (\Gamma_X)^G = \Gamma_Y(f_*^G)$. This gives two spectral functors converging to $H^n(X; G, A)$, whose second terms are respectively equal to $I_2^{pq} = H^p(Y, \mathcal{H}^q(G, A))$ and $II_2^{pq} = H^p(G, H^q(X, A))$. Suppose three G-bundles A, B, C and a $\cup$-product $A \otimes B \to C$, compatible with the G-structures on A, B, C, are given. The product structure can be extended in a natural way to the derived functors of the functors Γ^G and f_*^G. This gives mappings

$$H^p(X; G, A) \otimes H^q(X; G, B) \to H^{p+q}(X; G, C)$$

and

$$\mathcal{H}^p(G, A) \otimes \mathcal{H}^q(G, B) \to \mathcal{H}^{p+q}(G, C),$$

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possessing the usual properties of products. In what follows we shall be especially interested in the case when G is discrete on the space X; in this case, for any point $y \in Y$,

$$\mathcal{H}^p(G, A)_{(y)} \overset{\theta_y}{\approx} H^p(G_x, A), \tag{2}$$

where $x \in f^{-1}(y)$, and G_x is the stabilizer of the point x. This isomorphism carries the product structure in $\mathcal{H}^*(G,)_{(y)}$ to the ordinary multiplication in the cohomology groups of groups.

The product $A \otimes B \to C$ is also carried in a natural way to the spectral sequences I and II. Namely, mappings

$$I_r^{pq}(A) \otimes I_r^{p'q'}(B) \to I_r^{p+p',q+q'}(C)$$

are defined, compatible with the structures of differential groups on I_r and compatible with the product structure

in the limiting term. Moreover, on I_2 this product coincides, up to sign, with the natural product present in $H^*(Y, \mathcal{H}^*(G))$. The situation is analogous for the second spectral sequence as well.

Let us also note the following naturality property of the functors under consideration. Let Π be a subgroup of the group G; let Y' be the quotient space of X by Π; and let I' and II' be the spectral sequences I and II, considered with respect to Π. The inclusion $\Pi \subset G$ induces a homomorphism of spectral sequences $I \to I'$ and $II \to II'$, which on I_2 and II_2 coincides with the natural homomorphisms, for example on II_2 with the restriction homomorphism.

2. We return to the proof of (1). Thus, $\mathfrak{A}$ is a division algebra, $J = [\tau_1, \dots, \tau_n]$ is its maximal order; G is the group of proper units in J (norm $+1$). Denote by $x \to A_x$, $x \in \mathfrak{A}$, the left regular representation of the algebra $\mathfrak{A}$ in the basis $\tau_1, \dots, \tau_n$. Since $\mathfrak{A}$ has no zero divisors, all matrices A_x, $x \in \mathfrak{A}$, $x \neq 1$, do not have $+1$ as an eigenvalue and, in particular, the group G acts without fixed points on the space $X = \mathfrak{A} \otimes R - (0)$ (R is the field of real numbers).

Lemma. Let π be a finite subgroup of order s of the group G; then

$$H^n(\pi, Z) = Z/sZ.$$

Moreover, in $H^n(G, Z)$ there is an element ξ such that, for all finite subgroups $\pi \subset G$, $i(G, \pi)\xi$ is a generator of the cyclic group $H^n(\pi, Z)$ ($i(G, \pi)$ is the restriction homomorphism).

Let II and II' be the spectral sequences for the pairs (G, X) and (π, X) with constant sheaf of coefficients Z. The space X is a homological sphere S^{n-1}, and therefore in these spectral sequences there is only one nontrivial differential

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$d_n^{p,n-1} : II_n^{p,n-1} \to II_n^{p+n,0}$ ($d_n'^{p,n-1} : II_n'^{p,n-1} \to II_n'^{p+n,0}$). The first part of the lemma follows from the fact that the group π is discrete on X (since it is finite), and the transformations from G preserve the orientation of the space X; moreover $d_n'^{0,n-1}(1)$ is a generating element for $H^n(\pi, Z)$. As the element $\xi \in H^n(G, Z)$ one may take $d_n^{0,n-1}(1)$; indeed, if we denote by h the homomorphism of spectral sequences $II \to II'$ induced by the inclusion $\pi \subset G$, then we shall have

$$hd_n^{0,n-1}(1) = d_n'^{0,n-1}(1),$$

while on $II_2^{p,0}$ h coincides with the restriction homomorphism.

Denote by $\smile: H^p(G,Z) \otimes H^q(G,A) \to H^{p+q}(G,A)$ the product induced by the product $Z \otimes A \to A$, $1 \otimes a \to a$.

Theorem 1. Let G be the group of proper units in $\mathfrak{A}$; let n be the rank of $\mathfrak{A}$ over the field of rational numbers Q; and let A be an arbitrary G-module. In the group $H^n(G, Z)$ there is an element ξ such that for any $k \geq n(n+1)/2$ multiplication by ξ is an isomorphism

$$H^k(G, A) \approx H^{n+k}(G, A).$$

Proof. We shall prove the following more general assertion. Let G be a discrete subgroup of the group $GL(n, R)$, consisting of matrices of determinant $+1$, and suppose that the assertion of the lemma holds for G; then Theorem 1 is true for G.

Denote by $\mathfrak{P}_n$ the space of real positive quadratic forms in n variables, and by X the discriminant surface in $\mathfrak{P}_n$, i.e. the set of $f \in \mathfrak{P}_n$ with $\det f = 1$. From our assumptions on G it follows that G acts discretely on X. Denote

by Z and A the constant sheaves over X with stalks Z and A, and consider the spectral sequences $I(Z)$ and $I(A)$ for the pair (G, X). First of all it is clear that they converge to $H^*(G, Z)$ and $H^*(G, A)$, since the space X is contractible to a point. The homomorphism $\varphi : H^q(G,Z) \to \Gamma_Y(\mathcal{H}^q(G,Z))$, obtained from the spectral sequence I, is constructed as follows: if $\eta \in H^q(G, Z)$, then for any point $y \in Y$, φ on the stalk over the point y coincides with the restriction homomorphism, if $\mathcal{H}^q(G, Z)_{(y)}$ is identified with $H^q(G_x, Z)$, $x \in f^{-1}(y)$, by means of the isomorphism (2),

$$(\varphi\eta)(y) = \theta_y^{-1} i(G, G_x)\eta.$$

Thus, if $\eta \in H^q(G, Z)$, then the function $\psi(\eta) : y \mapsto \theta_y^{-1} i(G, G_x)\eta$ belongs to $I_2^{0,q}(Z)$ and $d_r^{0,q}\psi = 0$ for $r = 2, 3, \ldots$. Let $\xi \in H^n(G, Z)$ be an element satisfying the conclusion of the lemma; $\psi(\xi) \in I_r^{0,n}(Z)$, $r \geq 2$; let us see how the

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spectral sequence $I(A)$ behaves under multiplication by $\psi(\xi)$. We shall prove that multiplication by $\psi(\xi)$ is an isomorphism

$$I_r^{pq}(A) \longrightarrow I_r^{p,q+n}(A), \qquad q \geq \frac{n(n+1)}{2} \tag{3}$$

for all $r \geq 2$. Since $d_r^{0,n}\psi(\xi) = 0$ for $r \geq 2$, it is enough to prove this for $r = 2$. In this case the mapping (3) becomes the homomorphism

$$H^p(Y, \mathcal{H}^q(G, A)) \longrightarrow H^p(Y, \mathcal{H}^{q+n}(G, A)),$$

induced by multiplication of the coefficients by $\psi(\xi)$; therefore it suffices to prove that multiplication by $\psi(\xi)$ is an isomorphism of sheaves

$$\mathcal{H}^q(G, A) \longrightarrow \mathcal{H}^{q+n}(G, A), \qquad q > 0.$$

Let $y \in Y$; then the transformation θ_y carries α_y into multiplication by $i(G, G_x)\xi$,

$$H^q(G_x, A) \xrightarrow{\theta_y \alpha_y \theta_y^{-1}} H^{q+n}(G_x, A), \qquad q > 0.$$

By the lemma, G_x is a finite group with $H^n(G_x, Z) = Z \backslash (\operatorname{ord} G_x)Z$, but then G_x is a periodic group in the sense of Cartan-Eilenberg[2], and since $i(G, G_x)\xi$ is a generator of $H^n(G_x, Z)$, it follows from [2] that multiplication by this element is an isomorphism for any module A. Thus (3) is an isomorphism.

The space Y is a cell complex of dimension $n(n+1)/2 - 1$; hence $I_\infty^{p,q} = 0$ for $p \geq n(n+1)/2$. Together with the isomorphisms (3) for $r = \infty$, this shows that for $k \geq n(n+1)/2$ multiplication by ξ is an isomorphism of the groups associated with the filtered groups $H^k(G, A)$ and $H^{k+n}(G, A)$, and consequently these groups themselves are isomorphic. The theorem is proved.

Remark. It follows from the proof that $H^k(G, A) = 0$ for $k \geq n(n+1)/2$, if A is a vector space over a field of characteristic 0 (or relatively prime to the orders of the finite subgroups of the group G).

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REFERENCES

[1] A. Grothendieck, *Tohoku Math. J.*, **9**, 2/3, 119 (1957)
[2] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton, 1956.

Note: Figure translations are in progress. See original paper for figures.

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