Mathematics

L. G. MIKHAILOV

1961

SovietRxiv

Abstract

Full Text

Mathematics

L. G. MIKHAILOV

ELLIPTIC EQUATIONS WITH SINGULAR COEFFICIENTS

(Presented by Academician I. N. Vekua, March 4, 1961)

1. Notation. E is n-dimensional Euclidean space; $x = (x_1, x_2, \dots, x_n)$; $f(x) = f(x_1, x_2, \dots, x_n)$, $dx = dx_1, dx_2, \dots, dx_n$, $r(x,t) = \left(\sum_{i=1}^{n}(x_i - t_i)^2\right)^{1/2}$, $\rho(x) = r(x, 0)$; $S(D)$ is the class of bounded measurable functions with norm $\|f\|_{S(D)} = \sup_{x\in D}|f(x)|$, $S(\beta, D)$ is the class of functions representable in the form $f(x) = \rho^{-\beta}(x)f_0(x)$, where $f_0(x) \in S(D)$. If the norm $\|f\|_{S(\beta,D)} = \|f_0\|_{S(D)}$ is introduced, then $S(\beta, D)$ will be isometric to $S(D)$.

Let there be p distinct points $c_1, c_2, \dots, c_p$ inside and on the boundary of the domain D, and let $\rho_i(x) = r(x, c_i)$. By $\Pi(x)$ we denote the distance from the point x to the set $c_1, c_2, \dots, c_p$, i.e.

$$\Pi(x) = \min\{\rho_1(x), \rho_2(x), \dots, \rho_p(x)\};$$

$S(\beta, \Pi, D)$ is the class of functions of the form $f(x) = \Pi^{-\beta}(x)f_0(x)$, where $f_0(x) \in S(D)$, with norm $\|f\|_{S(\beta,\Pi,D)} = \|f_0\|_{S(D)}$; K is a cone with vertex at the point 0 and infinite in one direction; $q(K, \alpha, \beta) = \int_K \rho^{-\beta}(y) r^{n-\alpha}(y, I)\, dy$, where $0 < \alpha < n$, $\alpha < \beta < n$; I is a point of the unit sphere, $I \in K$. If $K = E$, then we shall write $q(E, \alpha, \beta) = q(\alpha, \beta)$.

2. A new class of integral equations. Let D be an arbitrary bounded or unbounded domain. We first consider the simplest operator

$$T\varphi = \rho^{-\alpha}(x) \int_D r^{\alpha-n}(x,t)\varphi(t)\, dt, \tag{1}$$

where $0 < \alpha < n$ and 0 is an interior point of the domain D.

Theorem 1. *Formula (1) defines a linear (not completely continuous) operator in the Banach spaces $S(\beta, D)$, $\alpha < \beta < n$, and*

$$\|T\|_{S(\beta,D)} = q(\alpha, \beta).$$

Generalization. Let there be several singular points $c_1, c_2, \dots, c_p$, which may lie both inside and on the boundary of the domain. If the domain is unbounded, then the infinitely distant point is also included among the singular points; it, in turn, may be interior or boundary and is denoted by c_{p+1}.

Then the formula

$$T_\Pi \varphi = \Pi^{-\alpha}(x) \int_D r^{\alpha-n}(x,t)\varphi(t)\,dt$$

defines a linear operator in $S(\beta,\Pi,D)$, $\alpha<\beta<n$, and

$$q(K_i,\alpha,\beta) \le \|T_\Pi\|_{S(\beta,\Pi,D)} \le Bq(V_i,\alpha,\beta), \qquad i=1,2,\dots,p+1,$$

where K_i, V_i are the cones of tangency and visibility of the domain D from the point c_i; the constant B does not depend on the domain.

If the point c_i is interior, we put $K_i = V_i = E$.

Theorem 2. If $K(x,t) \in S(D\times D)$, then the formula

$$K\varphi = \Pi^{-\alpha}(x) \int_D r^{\alpha-n}(x,t)K(x,t)\varphi(t)\,dt$$

defines a linear operator in $S(\beta,\Pi,D)$, $\alpha<\beta<n$, and

$$\mu_i q(K_i,\alpha,\beta) \le \|K\|_{S(\beta,\Pi,D)} \le Mq(V_i,\alpha,\beta), \tag{2}$$

where

$$M = \sup_{x,t\in D}|K(x,t)|, \qquad \mu_i = \lim_{x\to c_i}\lim_{t\to c_i}|K(x,t)|, \qquad i=1,2,\dots,p+1.$$

Theorem 3. If $K(x,t)$ is continuous for $x\ne t$ and $\overline{\lim}|K(x,t)| = 0$ at all singular points, then the operator $K\varphi$ is completely continuous in $S(\beta,\Pi,D)$, $\alpha<\beta<n$, for every β.

Theorems 1-3 make it possible to give certain criteria for the solvability of the equation (cf. ([8,6])).

$$\varphi(x) + \int_D \frac{K(x,t)}{\Pi^\alpha(x)\,r^{n-\alpha}(x,t)}\varphi(t)\,dt = f(x), \tag{3}$$

where $f(x)\in S(\beta,\Pi,D)$, $\beta<n$. For sufficiently small $K(x,t)$ the solution exists and is unique, but this cannot be achieved by making the domain D small. Under the conditions of Theorem 3 the Fredholm theorems are valid. However, from Theorems 1 and 3 there follows a more general assertion.

Theorem 4. Let $K(x,t)$ be continuous for $x\ne t$ and at all singular points. If

$$\sum_{i=1}^{p+1}|K(c_i,c_i)|\,q(V_i,\alpha,\beta) < 1, \tag{4}$$

then the Fredholm theorems are valid for equation (3).

3. The manifold of solutions. Consider the elliptic equation of second order

$$Lu \equiv \Delta u + \sum_{i=1}^{n} \frac{a_i(x)}{\rho(x)} u'_{x_i} + \frac{b(x)}{\rho^2(x)} u = 0. \tag{5}$$

The point $x = 0$ (as well as $x = \infty$) is called its **regular singular point** if the functions $a_i(x)$, $i = 1, 2, \dots, n$, $b(x)$ are bounded at this point, and a **weak singularity** if, as $\rho \to 0$, $a_i(x), b(x) = O(\rho^{\varepsilon})$, $\varepsilon > 0$ (as $\rho \to \infty$, $O(\rho^{-\varepsilon})$).

To study the manifold of solutions we put

$$u(x) = -\int_D \psi(r)\varphi(t)\, dt + h(x), \tag{6}$$

where $\psi(r)$ is the fundamental solution of Laplace' s equation and $h(x)$ is an arbitrary function. Substituting (6) into (5), we obtain for $\varphi(x)$ an integral equation of the form (3), where $f(x)$ is arbitrary, $K = K_1 + K_2$, and *

$$K_1(x,t) = -\frac{1}{\omega_n} \frac{\sum_{i=1}^{n} r'_{x_i} a_i(x)}{\rho(x)\, r^{n-1}(x,t)}, \qquad K_2(x,t) = \frac{b(x)}{\omega_n (n-2)} \frac{1}{\rho^2(x)\, r^{n-2}(x,t)}. \tag{7}$$

* Henceforth the case $n = 2$, $b(0) \neq 0$ is everywhere excluded.

For a local investigation of a regular singular point, take as D a sufficiently small neighborhood of it and pass from $x = 0$ to $x = \infty$. Then Theorem 2 immediately implies:

Theorem 5. *If* $\lim_{x\to 0} a_i(x)$, $i = 1, \dots, n$, *and* $\lim_{x \to 0} b(x)$ *are sufficiently small, then there always exist solutions continuous at the singular point.*

Consider equations with two regular singular points. By translations and inversions we can always move them to the points $0, \infty$, so that the equation takes the form (5), where $a_i(x), b(x)$ are defined and bounded in the whole space E. In this case, and also for the case $p > 2$ of singular points, we put $D = E$ in (6). From Theorems 2 and 4 we obtain a number of results, of which we shall present one that has some connection with Schrödinger' s problem known in quantum mechanics:

Theorem 6. *If, of two regular singular points, one is a weak singularity, and at the other* $a_i(x), b(x)$ *are continuous and small, then equation (5) has a continuum of solutions continuous in the whole space, including both singular points.*

4. The first boundary-value problem. Let D be a bounded domain whose boundary consists of a finite number of Lyapunov surfaces. If $G(x,t)$ is the Green's function of the Laplace operator, then we shall also require that (cf. (1^2))

$$|G'_{x_i}(x,t)| \le Hr^{1-n}, \qquad i = 1, \dots, n. \tag{8}$$

The formulation of the first boundary-value problem is: for the given values $u(x) = \psi(x)$ on the boundary of the domain, find solutions of the equation $Lu = \rho^{-2}(x)g(x)$, $g(x) \in S(D)$, regular everywhere in D except at the point $x = 0$, and admitting at this point a singularity whose order is restricted by the condition $\Delta u \in S(\beta, D)$, $\beta < n$.

As usual, reducing the boundary values to zero and setting

$$u(x) = -\int_D G(x,t)\varphi(t)\,dt,$$

we obtain for $\varphi(x)$ the integral equation (3) with kernels of type (7) (see (9)).

Theorem 7. *Let $b(x) = O(\rho^\varepsilon)$, $\varepsilon > 0$, $n \ge 2$. If the $a_i(x)$, $i = 1, \dots, n$, and either $b(x)$ or the domain D are sufficiently small, then the boundary-value problem has a unique solution, continuous at the singular point.*

If the number β is fixed by the conditions $2 < \beta < n$,

$$Hq(1,\beta)\sum_{i=1}^{n} \sup_{x\in D}|a_i(x)| + \frac{q(2,\beta)}{\omega_n(n-2)} \sup_{x\in D}|b(x)| < 1,$$

then the boundary-value problem has a unique solution $u(x) \in S(\beta - 2, D)$.

Theorem 8. *If $a_i(x)$, $i = 1, 2, \dots, n$, and $b(x)$ are continuous at the singular point and*

$$Hq(1,\beta)\sum_{i=1}^{n} |a_i(0)| + \frac{q(2,\beta)}{\omega_n(n-2)}|b(0)| < 1$$

(and for $n = 2$, moreover, $b(x) = O(\rho^\varepsilon)$, $\varepsilon > 0$), then the Fredholm theorems hold for the first boundary-value problem.

Theorem 9. *If $b(x) \le 0$ in the domain D, then under the conditions of Theorem 8 the boundary-value problem is solvable uniquely and the solution is bounded at the singular point.*

The proof of this theorem is based on the generalized maximum principle (10).

Consider the equation

$$\Delta u + \sum_1^n A_i(x) u'_{x_i} + B(x) u = F(x) \qquad (9)$$

with several regular singular points $c_1, c_2, \dots, c_p$ and $c_{p+1} = \infty$, which may lie both inside and on the boundary of the domain, being a point of smoothness or a conical point of the surface. In other words, let $A_i(x) \in S(1, \Pi, D)$, $i = 1, \dots, n$; $B(x), F(x) \in S(\alpha, \Pi, D)$, $\alpha \leq 2$, where $\Pi(x) = \min_{1 \leq i \leq p}\{r(x, c_i)\}$. Suppose that the Green' s function of the Laplace equation still satisfies conditions (8).

Then theorems of the type 7, 8, 9 hold. Let us note only that if the singular points lie on the boundary of the domain, then in theorem 9 one cannot assert uniqueness and boundedness of the solution. The results also extend to equations of noncanonical form.

The present work is a further development of the author' s papers ([6,7]), in which the generalized Cauchy–Riemann system and the integral equations associated with it were studied. Singularities below the first order and somewhat more general classes of coefficients were considered in ([4,5]). If (5) is multiplied by $\rho^2(x)$, then one may speak of degeneration of the order of the equation. Degenerations of various types have been the subject of many works; see, for example, ([11,12]). However, all these works do not cover equations (5).

Department of Physics and Mathematics
Academy of Sciences of the Tajik SSR

Received
25 II 1961

REFERENCES

1. V. I. Smirnov, *A Course of Higher Mathematics*, 4, Moscow–Leningrad, 1951.
2. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Moscow–Leningrad, 1951.
3. C. Miranda, *Partial Differential Equations of Elliptic Type*, Moscow, 1957.
4. G. Giraud, C. R., 194, 1142 (1932).
5. I. N. Vekua, DAN, 101, No. 4 (1955).
6. L. G. Mikhailov, DAN, 119, No. 1 (1958).
7. L. G. Mikhailov, DAN, 129, No. 3 (1959).

8. L. G. Mikhailov, Izv. AN Tajik SSR, Division of Phys.-Math., Geological-Chemical Sciences, 1 (3) (1961).

9. L. G. Mikhailov, Dokl. AN Tajik SSR, 3, No. 5 (1960).

10. A. I. Achil' diev, Dokl. AN Tajik SSR, 4, No. 1 (1961).

11. V. K. Zakharov, DAN, 124, No. 4 (1959).

12. V. P. Glushko, DAN, 129, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.