MATHEMATICS

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1961

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Abstract

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MATHEMATICS

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ON THE CONCEPT OF A DERIVED SYSTEM IN THE SENSE OF M. A. LAVRENT' EV

(Presented by Academician M. A. Lavrent' ev, 3 X 1960)

1°. In the works ([1], [2]) M. A. Lavrent' ev introduced the following concept of a derived system for a given (generally speaking, nonlinear) system of partial differential equations

$$F_i(x, y, u, v, u_x, u_y, v_x, v_y) = 0 \quad (i = 1, 2), \tag{1}$$

which generalizes the concept of the hodograph system of equations introduced by S. A. Chaplygin. At any point $z_0 = x_0 + iy_0$ at which the Jacobian of the mapping effected by a solution of system (1) is nonzero, the principal linear part of such a mapping transforms the unit square, whose base is inclined at an angle β to the u-axis, into a parallelogram with base V_β, inclined to the x-axis at an angle α_β, height W_β, and angle at the vertex θ_β. The quantities $V_\beta, W_\beta, \alpha_\beta, \theta_\beta$ are called the **characteristics** of the mapping; in terms of them the partial derivatives of the solution are expressed elementarily:

$$u_x = \left(\frac{\operatorname{tg}\alpha + \operatorname{tg}\theta}{V_\beta \operatorname{tg}\theta} + \frac{\operatorname{tg}\alpha\ \operatorname{tg}\beta}{W_\beta}\right)\cos\alpha\cos\beta, \qquad u_y = \left(\frac{\operatorname{tg}\alpha\ \operatorname{tg}\theta - 1}{V_\beta \operatorname{tg}\theta} - \frac{\operatorname{tg}\beta}{W_\beta}\right)\cos\alpha\cos\beta,$$

$$v_x = \left(\frac{\operatorname{tg}\alpha + \operatorname{tg}\theta}{V_\beta \operatorname{tg}\theta}\operatorname{tg}\beta - \frac{\operatorname{tg}\alpha}{W_\beta}\right)\cos\alpha\cos\beta, \qquad v_y = \left(\frac{\operatorname{tg}\alpha\ \operatorname{tg}\theta - 1}{V_\beta \operatorname{tg}\theta}\operatorname{tg}\beta + \frac{1}{W_\beta}\right)\cos\alpha\cos\beta. \tag{2}$$

Substituting these expressions into (1), we obtain the so-called **equations in characteristics**; we shall assume that the latter are solved with respect to W_β and θ_β:

$$W_\beta = W_\beta(x, y, u, v, V_\beta, \alpha_\beta), \qquad \theta_\beta = \theta_\beta(x, y, u, v, V_\beta, \alpha_\beta). \tag{3}$$

For $\beta = 0$ we shall denote the characteristics by the same letters, but without the subscript; relations (3) for $\beta = 0$ will be called the **basic equations in characteristics**:

$$W = W(x, y, u, v, V, \alpha), \qquad \theta = \theta(x, y, u, v, V, \alpha). \tag{4}$$

In the work of M. A. Lavrent' ev ([1]) it is proved that, for any solution of system (1), the quantities $P = \ln V$ and α satisfy the system of differential equations

$$\frac{\partial P}{\partial v} = a_1 \frac{\partial P}{\partial u} + a_2 \frac{\partial \alpha}{\partial u} + a_3, \qquad \frac{\partial \alpha}{\partial v} = b_1 \frac{\partial P}{\partial u} + b_2 \frac{\partial \alpha}{\partial u} + b_3, \tag{5}$$

which is called the **derived system** of system (1), and whose coefficients are expressed in terms of the right-hand sides of the basic equations in characteristics (4) by the formulas

$$a_1 = \frac{\partial W}{\partial V} \operatorname{ctg}\theta - \frac{W}{\sin^2\theta}\frac{\partial \theta}{\partial V}, \qquad a_2 = \frac{1}{V}\left(\frac{\partial W}{\partial \alpha}\operatorname{ctg}\theta - \frac{W}{\sin^2\theta}\frac{\partial\theta}{\partial\alpha} - W\right),$$
$$b_1 = \frac{\partial W}{\partial V}, \qquad b_2 = \frac{1}{V}\left(\frac{\partial W}{\partial \alpha} + W\operatorname{ctg}\theta\right),$$
$$a_3 = \left(\frac{1}{V}\frac{\partial W}{\partial u} + \frac{\partial W}{\partial s}\right)\operatorname{ctg}\theta - \left(\frac{1}{V}\frac{\partial \theta}{\partial u} + \frac{\partial\theta}{\partial s}\right)\frac{W}{\sin^2\theta}, \qquad b_3 = \frac{1}{V}\frac{\partial W}{\partial u} + \frac{\partial W}{\partial s} \tag{6}$$

$\left(\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha \text{ is the derivative in the direction of the base of the characteristic parallelogram}\right).$

For the case of the Cauchy–Riemann system (for which the basic equations in characteristics have the form $W = V,\ \theta = \pi/2$) the derived system $\partial P/\partial v = -\partial\alpha/\partial v,\ \partial\alpha/\partial v = \partial P/\partial u$ coincides with the basic one. For the case of the system of gas dynamics (for which $W = W(V),\ \theta = \pi/2$) the derived system

$$\frac{\partial P}{\partial v} = -\frac{W}{V}\frac{\partial\alpha}{\partial u}, \qquad \frac{\partial\alpha}{\partial v} = W'(V)\frac{\partial P}{\partial u}$$

coincides with S. A. Chaplygin' s system of hodograph equations. On this basis, also in the general case, we shall call the plane $\frac{1}{V}e^{-i\alpha}$ the **hodograph plane**.

2°. In applications the principal role is played by a theorem of M. A. Lavrent' ev ([1]) on the relation between the type of a system and its derived system. We shall give here a new, simpler proof of this theorem, introducing into it a certain refinement.

Theorem of M. A. Lavrent' ev. *For systems (1) with continuously differentiable right-hand sides (4) in some domain, except for the case of systems for which*

$$\frac{\partial W_\beta}{\partial V_\beta} = \frac{V^2}{V_\beta^2}\frac{\partial W}{\partial V}, \tag{7}$$

the positivity of the derivative $\partial W_\beta/\partial V_\beta$ for any $\beta,\ 0 \le \beta < 2\pi$, is equivalent to the ellipticity of the derived system (5) in the classical sense and to the condition $b_1 = \partial W/\partial V > 0$.

For the proof we shall obtain a relation expressing $\partial W_\beta/\partial V_\beta$ in terms of the right-hand sides (4) and their derivatives. To this end we use the expressions of the characteristics for arbitrary β through the characteristics for $\beta = 0$:

$$W_\beta = \frac{VW}{V_\beta} = \frac{VW\sin\theta}{\sqrt{V^2\cos^2\beta\sin^2\theta + 2VW\cos\beta\sin\beta\cos\theta\sin\theta + W^2\sin^2\beta}},$$

$$\operatorname{tg}\theta_\beta = \frac{2VW\sin^2\theta}{(W^2 - V^2\sin^2\theta)\sin 2\beta + VW\sin 2\theta\cos 2\beta}, \tag{8}$$

$$\operatorname{tg}\alpha_\beta = \frac{V\operatorname{tg}\alpha + W(1+\operatorname{tg}\alpha\operatorname{ctg}\theta)\operatorname{tg}\beta}{V - W(\operatorname{tg}\alpha - \operatorname{ctg}\theta)\operatorname{tg}\beta}.$$

Suppose that in the expressions (8) for V_β and α_β the W and θ from (4) have been substituted, and that these expressions have been solved with respect to V and α:

$$V = g_1(V_\beta, \alpha_\beta), \qquad \alpha = g_2(V_\beta, \alpha_\beta), \tag{9}$$

(in all that follows we shall assume that the values of the coordinates x, y, u, v are fixed, and shall not take dependence on them into account). We write formulas (8) briefly in the form

$$W_\beta = f_1(W, V, \theta), \qquad V_\beta = f_2(W, V, \theta), \qquad \alpha_\beta = f_3(W, V, \theta, \alpha).$$

Substituting into these formulas W and θ from (4) and V and α from (9), we find, by the rules for differentiating composite functions,

$$\frac{\partial W_\beta}{\partial V_\beta} = \frac{1}{V_\beta^2}\,\frac{V^3WVc^2 + VW\left[(W+VW_V)\operatorname{ctg}\theta + W_\alpha + \frac{VW}{\sin^2\theta}\theta_V\right]cs + \frac{W^3}{\sin^2\theta}(1+\theta_\alpha + V\theta_V\operatorname{ctg}\theta)s^2}{Vc^2 + \left(W\operatorname{ctg}\theta + W_\alpha + VW_V\operatorname{ctg}\theta - \frac{VW}{\sin^2\theta}\theta_V\right)cs + \frac{W}{\sin^2\theta}\left(\frac{\partial(W,\theta)}{\partial(V,\alpha)} + W_V - W\theta_V\operatorname{ctg}\theta\right)}, \tag{10}$$

where, for brevity, we have denoted $\cos\beta = c,\ \sin\beta = s$, and by subscripts—the corresponding partial derivatives.

Let the derived system (5) be elliptic, i.e., in the domain under consideration the inequality

$$K = a_2 b_1 + \left(\frac{a_1 - b_2}{2}\right)^2 < 0 \tag{11}$$

is satisfied.

A calculation shows that the discriminants of the quadratic forms (with respect to the variables $c = \cos\beta$ and $s = \sin\beta$) in the numerator and denominator of (10) are respectively equal to $-V^4 W^2 K$ and $-V^2 K$. Therefore, under the assumption made, both forms are definite; moreover, the form in the denominator is always positive, while the form in the numerator is positive by virtue of the condition $\partial W/\partial V = b_1 > 0$; consequently $\partial W_\beta/\partial V_\beta > 0$ for any β.

Conversely, if $\partial W_\beta/\partial V_\beta > 0$ for any β and the exceptional case (7) does not occur, i.e. the forms in the numerator and denominator of (10) are not proportional, then both of these forms are positive definite. But this means that the derived system is elliptic and $b_1 = \partial W/\partial V > 0$. The theorem is proved.

3°. **Remark.** The exceptional case mentioned in the theorem can in fact occur. Condition (7) is written in the form of two relations

$$(W_\alpha + W \operatorname{ctg}\theta + V W_V \operatorname{ctg}\theta)(W - V W_V) + \frac{VW}{\sin^2\theta}(W + V W_V)\theta_V = 0,$$

$$(1 + \theta_\alpha + V\theta_V \operatorname{ctg}\theta)(W^2 - V^2 W_V^2) + (W_\alpha + W \operatorname{ctg}\theta + V W_V \operatorname{ctg}\theta)V^2 W \theta_V = 0. \tag{12}$$

For the particular case of systems for which $\theta_V \equiv \theta_\alpha \equiv 0$, these relations reduce to $W \pm V W_V = 0$; moreover, the condition of positivity of $\partial W_\beta/\partial V_\beta$ can be satisfied only in the case $W = V W_V$, when the first basic equation in characteristics is linear with respect to V: $W = C(x, y, u, v, \alpha)V$.

Here is an example of such a system:

$$v_y = e^{-2c\alpha} u_x, \qquad v_x = -e^{-2c\alpha} u_y, \tag{13}$$

where c is a constant. The basic equations in characteristics for it have the form: $W = e^{2c\alpha} V$, $\theta \equiv \pi/2$, and

$$\frac{\partial W_\beta}{\partial V_\beta} = \frac{V^2}{V_\beta^2} e^{2c\alpha} > 0$$

for any c, while the derived system

$$\partial P/\partial v = -e^{2c\alpha}\partial\alpha/\partial u, \qquad \partial\alpha/\partial v = e^{2c\alpha}\partial P/\partial u + 2ce^{2c\alpha}\partial\alpha/\partial u$$

is elliptic only for $|c| < 1$.

4°. The method of the derived system, which has found important applications (1^{-4}), apparently is still far from exhausted. Its significance lies in the fact that it reduces the investigation of nonlinear systems (1) to the investigation of quasilinear systems (5). We give one more example of an application of this method.

Theorem. Let, for a system (1) not containing the coordinates explicitly, the right-hand sides of the basic equations in characteristics be twice continuously differentiable in the entire open hodograph plane $\frac{1}{V}e^{-i\alpha}$. If the derived system is everywhere elliptic and

$$\int_0 \frac{dt}{tm(t)} = \infty, \tag{14}$$

where

$$m(t) = \max_{V=t} \frac{a_2}{K}\left[1 - K + \left(\frac{a_1 + b_2}{2}\right)^2\right],$$

then system (1) has a solution mapping the domain of variation of the variable z onto the whole (open) w-plane.

For the proof we rewrite the derivative system, introducing, instead of V and α, the Cartesian coordinates of the hodograph plane $\sigma = \cos\alpha/V$, $\omega = -\sin\alpha/V$, after which this system takes the form

$$\partial\sigma/\partial v = A_1\partial\sigma/\partial u + A_2\partial\omega/\partial u, \quad \partial\omega/\partial v = B_1\partial\sigma/\partial u + B_2\partial\omega/\partial u,$$

where the coefficients are expressed in a known way in terms of σ and ω. For a solution with positive Jacobian $\partial(u,v)/\partial(\sigma,\omega)$, the latter system can be inverted, and we obtain a linear elliptic system

$$\partial v/\partial\omega = a\,\partial u/\partial\sigma + b\,\partial u/\partial\omega, \quad -\partial v/\partial\sigma = d\,\partial u/\partial\sigma + c\,\partial u/\partial\omega. \tag{15}$$

Using a method that somewhat generalizes the method of K. Andreian Cazacu (^{5}), we find a condition under which the hodograph plane is transformed into the whole w-plane. Expressing this condition in terms of the coefficients of the derivative system, we obtain (14).

If condition (14) is fulfilled, then we can construct a mapping $u = u(\sigma,\omega)$, $v = v(\sigma,\omega)$, continuously differentiable and with positive Jacobian. Inverting it, we find a continuously differentiable mapping in the whole w-plane, $\sigma = \sigma(u,v)$, $\omega = \omega(u,v)$. Thus we determine V and α, and by (4) also W and θ in the whole w-plane. Formulas (2) for $\beta = 0$, inverted for the derivatives x_u, x_v, y_u, y_v, make it possible to find these derivatives in the whole w-plane, and then the mapping $z = z(w)$, inverse to the desired one, is determined. The theorem is proved.

Remark. For systems defining "orthogonal" mappings, for which the basic equations in characteristics have the form $W = W(V, \alpha)$, $\theta \equiv \pi/2$, condition (14) takes the form:

$$\int_0 \frac{dt}{\max_{V=t} W \left[1 - \frac{1}{K}\left(\frac{W_\alpha^2}{4V^2} + 1\right)\right]} = \infty;$$

in particular, for systems of the type of the system of gas dynamics, for which also $W = W(V)$, it has the form

$$\int_0 \frac{dV}{W(V) + \dfrac{V}{W'(V)}} = \infty.$$

The last condition is certainly fulfilled for systems that are strongly elliptic in the sense of M. A. Lavrent' ev, for which $W'(V)$ and W/V are bounded above and below by positive constants ([4]).

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Received
23 VIII 1960

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Note: Figure translations are in progress. See original paper for figures.