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Abstract

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THE FUNDAMENTAL PROBLEM OF NON-RIEMANNIAN GEOMETRY IN THE BINARY DOMAIN

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1. Eisenhart and Veblen posed the problem of finding conditions under which a parallel displacement given by its $\Gamma^k_{ij}(x)$ is Riemannian ([1]). This integral problem of tensor analysis, also called the fundamental problem of non-Riemannian geometry, as well as the question of the reducibility of a space A_n to a Weyl space (W_n), served as a topic of investigation for the Princeton school of geometers. However, their results, set forth in Thomas' s monograph ([2]), do not exhaust the problem of metrizability of manifolds with symmetric affine connection A_n. Moreover, attempts to overcome it even for $n = 2$ are erroneous: the conditions for the reducibility of A_2 to W_2 in article ([3]), contrary to the assertion of its author, are only sufficient; in articles ([4,5]), conversely, requirements are listed which are only necessary for the reducibility of A_2 to a Riemannian space V_2.

In the present note a solution is proposed for the fundamental problem of non-Riemannian geometry in the binary domain.

2. As a preparatory step toward the problem of the present communication, we classify the spaces A_2 by their differential invariants, understanding the question of classification itself in the sense of partitioning the set of all possible objects of symmetric connection $\Gamma^k_{ij}(x)$ into nonequivalent nonempty classes without common elements. The latter reduces to finding the types of the affine-curvature tensor $B^\alpha_{\beta\gamma\delta}$, since for $n = 2$ the quantities $\Gamma^k_{ij}(x)$ and B^i_{jkl} determine one another (([2]), §101). Putting

$$B^\alpha_\beta = \tfrac{1}{2} B^\alpha_{\beta\lambda\mu} e^{\lambda\mu}, \tag{1}$$

where $e^{\lambda\mu} = -e^{\mu\lambda}$, $B^\alpha_{\beta\gamma\delta}$ may be identified with a linear vector-function of the first kind

$$B = \begin{pmatrix} B^1_1 & B^1_2 \\ B^2_1 & B^2_2 \end{pmatrix}. \tag{2}$$

In view of this fact, with any A_2 there is associated an arithmetic invariant $[e_1 e_2]$—the Weierstrass characteristic of the vector-function B, corresponding to the curvature tensor of the space, which we shall call the characteristic of the manifold A_2. We agree to assign to one class spaces with equal characteristics, although they are not necessarily equivalent to one another.

Thus we obtain:

Theorem 1. *The spaces A_2, according to their characteristics* [11], [(11)], [2], *split into three nonempty nonequivalent classes.*

3. If in a space A_2 with coefficients of parallel displacement $\Gamma^k_{ij}(x)$ $(-\Gamma^k_{ji})$ there exists a nonsingular symmetric tensor g_{ij}

which is an integral of the differential system

$$\frac{\partial g_{ij}}{\partial x^k} - g_{\sigma j}\Gamma^{\sigma}_{ik} - g_{i\sigma}\Gamma^{\sigma}_{jk} = \omega_k g_{ij}, \tag{3}$$

where ω_k is a nonzero vector, then A_2 is said to be reducible to Weyl geometry (W_2). Consideration of the integrability conditions of system (3),

$$g_{\mu j}B^{\mu}{}_{ikl} + g_{i\mu}B^{\mu}{}_{jkl} - g_{ij}B^{\sigma}{}_{\sigma kl} = 0, \tag{4}$$

where $B^{\lambda}{}_{\beta\gamma\delta}$ are the components of the tensor of affine curvature of the object $\Gamma^k_{ij}(x)$, for spaces A_2 of nonsimple type immediately leads to the conclusion:

Lemma 1. *An A_2 of nonsimple type is irreducible to W_2.*

We shall call a coordinate net the **principal net** of the space A_2 if the directions tangent to the lines of each of its families at each point coincide with the vectors of the principal direction of the vector function B. Let the manifold of affine connection of simple type under consideration be referred to its principal net. Then the matrix B, as is known, assumes diagonal form, and relations (4) reduce to the following two equations:

$$g_{11}\cdot(B^1{}_{112} - B^2{}_{212}) = 0, \qquad g_{22}\cdot(B^2{}_{212} - B^1{}_{112}) = 0. \tag{5}$$

Equations (5) can be satisfied in two ways:

I. $B^1{}_{112} - B^2{}_{212} = 0$. In this case the characteristic of the space is expressed by the symbol [(11)], and system (3), being completely integrable, admits three solutions: g_{11}, g_{12}, g_{22}, functionally independent with respect to the three arbitrary constants contained in them ([6]). Consequently, we have:

Lemma 2. *A space A_2 with characteristic* [(11)] *is Weyl.*

II. $g_{11} = g_{22} = 0$. Substituting these values into (3), we obtain a mixed system for g_{12}:

$$g_{11}\Gamma^2_{11} = 0, \qquad g_{12}\Gamma^2_{12} = 0, \qquad g_{12}\Gamma^1_{12} = 0, \qquad g_{12}\Gamma^1_{22} = 0,$$

$$\frac{\partial g_{12}}{\partial x^1} - g_{12}(\Gamma^1_{11} + \Gamma^2_{12}) = \omega_1 g_{12}, \tag{6}$$

$$\frac{\partial g_{12}}{\partial x^2} - g_{12}(\Gamma^1_{12} + \Gamma^2_{22}) = \omega_2 g_{12}.$$

Thus, in order that the mixed system (3), II have a solution g_{ij} subject to the restriction $|g_{ij}| \neq 0$, it is necessary that

$$\Gamma^2_{11} = \Gamma^1_{22} = \Gamma^1_{12} = \Gamma^2_{12} = 0. \tag{7}$$

Conversely, if conditions (7) are fulfilled, then g_{12} is determined by solving a completely integrable system. Its integral g_{12}, by the theorem on a mixed system of differential equations, depends on one arbitrary constant.

In terms of the theory of nets our result may be stated as follows:

Lemma 3. *A space A_2 with characteristic* [11] *is Weyl if and only if its principal net is simultaneously also a Descartes net.*

If the functions $\Gamma^k_{ij}(x)$ have first partial derivatives of order r, then A_2 is called of class $C^{(r)}$. Put $\varphi(\lambda) = |B - \lambda E|$. The rank of the matrix $\varphi'(B)$ will be denoted by ρ. Using the terms and notations introduced, the criterion for Weyl metrizability of two-dimensional spaces of symmetric affine connection may be formulated as follows:

Theorem 2. In order that a space A_2 of class $C^{(1)}$ be a Weyl space W_2, it is necessary and sufficient that one of the following two conditions hold:

a) $\varphi'(B) = 0$,

b) $[\varphi'(B)]^2 \neq 0$, $\rho = 2$, the principal net is Descartes.

4. As is known, the special case of Weyl spaces for which the complementary vectors ω_i are gradients is called Riemannian and is denoted by V_2. Taking this definition into account, from Theorem 2 we derive a criterion for Riemannian connectedness:

Theorem 3. In order that a space A_2 be Riemannian V_2, it is necessary and sufficient that either of the following two conditions hold:

a) $\varphi'(B) = 0$, $B^\sigma_\sigma = 0$;

b) $[\varphi'(B)]^2 \neq 0$, $\rho = 2$, $B^\sigma_\sigma = 0$, the principal net is Descartes.

Theorems 1 and 2 together exhaust the question of the metrizability of manifolds A_2 and provide a means of surveying the varieties of spaces of affine connection of two dimensions.

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Note: Figure translations are in progress. See original paper for figures.