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Abstract

Full Text

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ON THE ASYMPTOTIC POWER AND EFFICIENCY OF THE CRITERION ω_n^2

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§ 1.

Let $x_1, x_2, \ldots, x_n$ be a sample of n independent observations of a random variable with distribution function $F(x)$. We shall denote the empirical distribution function of such a sample by $F^{(n)}(x)$. The criterion ω_n^2 is a criterion for testing the simple hypothesis $F(x) = F_0(x)$, based on the statistic

$$\omega_n^2(F) = n \int_{-\infty}^{\infty} \left[F^{(n)}(x) - F_0(x)\right]^2 \psi(F_0(x))\, dF_0(x), \tag{1}$$

where $\psi(u) \geq 0$ $(0 \leq u \leq 1)$ is a certain weight function. We introduce the argument F to denote the true distribution of the sample. Under the assumption $F(x) = F_0(x)$, the limiting distribution of the quantity ω_n^2 as $n \to \infty$ was studied by N. V. Smirnov ([1]) and by Anderson and Darling ([2]).

Denote

$$W^{(n)}(x; F) = \mathbf{P}\{\omega_n^2(F) < x\}$$

and

$$P_\alpha^{(n)}(F) = 1 - W^{(n)}(x_\alpha; F),$$

where x_α is the root of the equation $W(x; F_0) = 1 - \alpha$. At significance level α, $P_\alpha^{(n)}(F)$ is the power function of the criterion (the probability of rejecting the hypothesis if the alternative $F(x)$ is true).

Denote

$$\rho_F^2 = \int_{-\infty}^{\infty} \left[F(x) - F_0(x)\right]^2 \psi(F_0(x))\, dF_0(x). \tag{2}$$

Let the function $\delta(u)$, $0 \leq u \leq 1$, be such that $\delta(0) = \delta(1) = 0$ and

$$\int_0^1 \delta^2(u)\psi(u)\, du = 1. \tag{3}$$

We shall call the class of functions of the form

$$F_a(x) = F_0(x) + \frac{a}{\sqrt{n}}\,\delta(F_0(x))$$

the class $[\delta(u)]$. By (3), $a^2 = n\rho^2_{F_a}$. For functions $F_a(x) \in [\delta(u)]$, we introduce the notation $\omega^2_n(a)$, $W^{(n)}(x,a)$, and $P^{(n)}_\alpha(a)$ instead of $\omega^2_n(F_a)$, $W^{(n)}(x;F_a)$, and $P^{(n)}_\alpha(F_a)$.

Denote by λ_j and $f_j(u)$ $(j = 1, 2, ...)$ the eigenvalues and eigenfunctions of the integral equation

$$f(u) = \lambda \int_0^1 K(u,v) f(v)\, dv,$$

where

$$K(u,v) = [\min(u,v) - uv]\sqrt{\psi(u)}\sqrt{\psi(v)}, \qquad 0 \le u, v \le 1.$$

The kernel $K(u,v)$ is positive definite, whence $\lambda_j > 0$ $(j = 1, 2, ...)$; we shall assume that $\lambda_k \ge \lambda_j$ for $k > j$. In addition, the system of eigenfunctions $\{f_j(u)\}$ may be chosen orthonormal:

$$\int_0^1 f_j(u) f_k(u)\, du = \delta_{jk}.$$

By $D(\lambda)$ we shall denote the Fredholm determinant of the integral equation.

§ 2. Suppose:

I. The functions $F_0(x)$ and $\delta(u)$ are continuous.

II. The function $\psi(u)$ is continuous in any interval $0 < u_1 \le u \le u_2 < 1$, and the integral

$$\int_0^1 u(1-u)\psi(u)\, du = \int_0^1 K(u,u)\, du$$

exists.

III. The integral

$$\int_0^1 \delta(u)\psi(u)\, du$$

exists.

IV. For the expansion

$$\delta(u)\sqrt{\psi(u)} = \sum_{k=1}^{\infty} \delta_k f_k(u),$$

where

$$\delta_k = \int_0^1 f_k(u)\delta(u)\sqrt{\psi(u)}\, du,$$

the closure condition is satisfied.

Theorem 1. *If conditions I-III are satisfied, $W^{(n)}(x,a)$, for each a, converges weakly as $n \to \infty$ to $W(x,a) = \mathbf{P}\{\omega^2(a) < x\}$, where*

$$\omega^2(a) = \int_0^1 [y(u) + a\delta(u)]^2 \psi(u)\, du, \tag{4}$$

$y(u)$, $0 \le u \le 1$, is a Gaussian random process with $\mathbf{M}y(u) = 0$, $\mathbf{M}y(u)y(v) = \min(u,v) - uv$.

Theorem 2. *Under conditions I-IV, $W(x,a)$ has the characteristic function*

$$\varphi(t,a) = \frac{1}{\sqrt{D(2it)}} \exp\left\{ a^2 \sum_{k=1}^{\infty} \frac{it\lambda_k \delta_k^2}{\lambda_k - 2it} \right\}. \tag{5}$$

Let us note the expansion from which (5) is obtained:

$$\omega^2(a) = \sum_{k=1}^{\infty} \left(\frac{X_k}{\sqrt{\lambda_k}} + a\delta_k \right)^2, \tag{6}$$

where $\{X_k\}$ are independent normally $(0,1)$ distributed quantities.

Theorem 3. *As $a \to \infty$,*

$$W(x,a) - \Phi\left(\frac{x - a^2}{2a\sigma}\right) \to 0 \tag{7}$$

uniformly in x. Here

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

and

$$\sigma^2 = \int_0^1 \int_0^1 K(u,v)\delta(u)\delta(v)\sqrt{\psi(u)}\sqrt{\psi(v)}\, du\, dv. \tag{8}$$

Proof. The quantity

$$\xi = \int_0^1 y(u)\delta(u)\psi(u)\, du$$

is normally distributed with parameters $(0,\sigma)$. From (4) and (3) we obtain

$$\begin{aligned} W(x,a) &= \mathbf{P}\{\omega^2(a) < x\} = \mathbf{P}\{\omega^2(0) + 2a\xi + a^2 < x\} \\ &= \mathbf{P}\left\{ \frac{\omega^2(0)}{2a} + \xi < \frac{x - a^2}{2a} \right\}, \end{aligned} \tag{9}$$

whence (7) follows.

Chapman [3] was the first to point out the validity of (7), proceeding from somewhat different considerations.

1. It can be shown that, for each x, $W^{(n)}(x,a) \to W(x,a)$ uniformly in a.
2. From (5) it follows that $W(x,a) = W(x,-a)$; we shall therefore assume $a \geqslant 0$.
3. From (6) it follows that, for each $x > 0$, the function $W(x,a)$ decreases monotonically (in a), since the distribution function of each of the terms has this property. Hence it follows that $P_\alpha(a)$ increases monotonically; in particular, $P_\alpha(a) > P_\alpha(0) = \alpha$, if $a > 0$. Thus, ω^2 is an asymptotically unbiased test.
4. From (9) it follows that, for all x and a,

$$W(x,a) < \Phi\left(\frac{x-a^2}{2a\sigma}\right),$$

which gives a lower bound for the power function:

$$P_\alpha(a) > 1 - \Phi\left(\frac{x_\alpha - a^2}{2a\sigma}\right).$$

5. From (8) and (3) we obtain the estimate

$$\sigma^2 = \sum_{k=1}^{\infty} \frac{\delta_k^2}{\lambda_k} < \frac{1}{\lambda_1}.$$

Now from (10) it follows that, for the family of alternatives $\{F(x)\}$ with

$$\rho_F \geqslant \frac{\sqrt{x_\alpha}}{\sqrt{n}},$$

$$P_\alpha(F) \geqslant 1 - \Phi\left(\frac{x_\alpha - n\rho_F^2}{2\rho_F\sqrt{n}}\sqrt{\lambda_1}\right)$$

uniformly for all distribution functions of the family. In particular, if a sequence of alternatives $\{F_n(x)\}$ is such that $\rho_{F_n}\sqrt{n} \to \infty$, then $P_\alpha(F_n) \to 1$.

6. For $a \leqslant x$

$$W(x,a) = \mathrm{P}\{\omega^2(0) + 2a\xi < x - a^2\} > \mathrm{P}\left\{\omega^2(0) < x - a^2 - 2a\frac{1}{\sqrt{a}}\right\} -$$

$$-\mathrm{P}\left\{\xi > \frac{1}{\sqrt{a}}\right\} \geqslant W(x - a^2 - 2\sqrt{a}, 0) - \left[1 - \Phi\left(\left(\frac{1}{\sqrt{a}} - \sqrt{x}\right)\sqrt{\lambda_1}\right)\right],$$

or, for $a \leqslant x_\alpha$,

$$P_\alpha(a) \leqslant 1 - W(x_\alpha - a^2 - 2\sqrt{a}, 0) - \left[1 - \Phi\left(\left(\frac{1}{\sqrt{a}} - \sqrt{x_\alpha}\right)\sqrt{\lambda_1}\right)\right].$$

Hence, if a sequence $\{F_n(x)\}$ is such that $\rho_{F_n} = o(1/\sqrt{n})$, then $P_\alpha(F_n) \to \alpha$.

7. From (5) the semi-invariants of $W(x, a)$ are easily found:

$$k_n(a) = 2^{n-1}(n-1)!\left(\sum_{j=1}^{\infty}\frac{1}{\lambda_j^n} + a^2 n\sum_{j=1}^{\infty}\frac{\delta_j^2}{\lambda_j^{n-1}}\right) = 2^{n-1}(n-1)!\left(\int_0^1 K_n(u,u)\,du + \right.$$

$$\left. + a^2 n\int_0^1\int_0^1 K_{n-1}(u,v)\delta(u)\delta(v)\sqrt{\psi(u)}\sqrt{\psi(v)}\,du\,dv\right),$$

where $K_n(u, v)$ is the n-th iteration of the kernel $K(u, v)$.

8. Put

$$\nu_n(x_\alpha) = \int_{+0}^{\infty} a^n\,dP_\alpha(a).$$

Using (5), one can find the Laplace transform of $\nu_n(x)$:

$$\tilde{\nu}_n(p) = \int_0^\infty e^{-px}\nu_n(x)\,dx = \Gamma(n/2+1)\left|p\sqrt{D(-2p)}\right|\left(\sum_{k=1}^{\infty}\frac{\lambda_k\delta_k^2 p}{\lambda_k + 2p}\right)^{n/2}.$$

§ 3. To compute the asymptotic efficiency of the criterion in the general case, we shall find, for the family of alternatives $[\delta(u)]$, the asymptotically most powerful criterion and its power function $P_\alpha^*(a)$. Then, if $P_\alpha^*(a^*) = P_\alpha(a) = 1 - \beta$, where β is the prescribed error of the second kind, the asymptotic efficiency of the criterion is equal to $e = a^{*2}/a^2$.

Under condition I, without loss of generality one may take $F_0(x) = x$, $0 \leq x \leq 1$. Then $F_a(x) = x + \dfrac{a}{\sqrt{n}}\delta(x)$.

Theorem 4. *Suppose that $\delta(u)$ is differentiable and that the integral*

$$s^2 = \int_0^1 [\delta'(u)]^2\,du$$

exists. Then, if $F(x) = F_a(x)$*, the quantity*

$$T^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta'(x_i)$$

is asymptotically normally distributed $(as^2,\ s)$*, and the criterion based on the statistic* $T^{(n)}$ *is the asymptotically most powerful criterion for testing the hypothesis* $F(x) = x$ *against the alternative* $F(x) = F_a(x)$*.*

Proof. Since $T^{(n)}$ is a sum of independent identically distributed terms, the first assertion follows directly from the central limit theorem. The second assertion follows from the fact that the criterion $T^{(n)}$ is asymptotically equivalent to the criterion based on the likelihood ratio

$$\sum_{i=1}^{n} \ln \left[1 + \frac{a}{\sqrt{n}} \delta'(x_i)\right].$$

Denote by z_α the root of the equation $1 - \Phi(z) = \alpha$. The power function of the criterion $T^{(n)}$ is asymptotically equal to $\Phi(|a|s - z_\alpha)$. Using (10), we obtain

$$e \geq \frac{z_\alpha + z_\beta}{s\left(\sigma z_\beta + \sqrt{\sigma^2 z_\beta^2 + \chi_\alpha}\right)}.$$

For example, for the criterion with $\psi \equiv 1$ under the hypothesis $\Phi(x)$ and the alternatives $\Phi(x - \mu)$ and $\Phi\left(\frac{x}{1+\vartheta}\right)$, the efficiencies ($e_\mu$ and e_ϑ, respectively) will be: for $\alpha = \beta = 0.05$, $e_\mu \geq 0.53$, $e_\vartheta \geq 0.18$; for $\alpha = \beta = 0.01$, $e_\mu \geq 0.56$, $e_\vartheta \geq 0.20$; for $\alpha = \beta = 0.001$, $e_\mu \geq 0.58$, $e_\vartheta \geq 0.21$.

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