ON THE CONVERGENCE OF CERTAIN DIFFERENCE SCHEMES

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Abstract

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In proving the convergence of difference schemes one usually uses either the existence, for the corresponding difference scheme, of a smooth solution of the problem for the differential equation, or the uniqueness of the solution of this problem. In the first case a direct estimate is made of the deviation of the solution of the difference scheme from the exact solution of the problem for the differential equation; in the second case, on the basis of a priori estimates of the solutions of the difference scheme, compactness of these solutions is established, and then from the uniqueness of the limiting function the convergence of the solutions of the difference scheme follows as the mesh sizes tend to zero.

In the present note the proof of convergence of solutions of difference schemes is carried out directly on the basis of one-sided a priori estimates, without using the theorem on existence or uniqueness of the solution of the problem for the differential equation. In this process a monotone sequence is constructed from the solutions of the difference scheme, and from its convergence the convergence of the difference scheme follows. Below we shall prove in this way the convergence of certain difference schemes. In the examples considered, the solutions of the corresponding problems for the differential equation are discontinuous or nonsmooth functions.

1. Lax' s scheme (1) for a quasilinear equation. We shall prove the convergence of the Lax difference scheme for the solution of the Cauchy problem for $t \geqslant 0$ for the simplest quasilinear equation

$$\partial u/\partial t + \partial\varphi(u)/\partial x = 0 \tag{1}$$

with the condition

$$u(0,x) = u_0(x), \quad -\infty < x < +\infty, \quad \text{where } |u_0(x)| \leqslant M. \tag{2}$$

The convergence of this scheme was first established in ([2]). The Lax difference scheme has the form:

$$u_n^{k+1} = \tfrac{1}{2}(u_{n+1}^k + u_{n-1}^k) - [\varphi(u_{n+1}^k) - \varphi(u_{n-1}^k)]h/2l, \tag{3}$$

where $u_n^k = u_{hl}(kh, nl)$, h and l are some positive numbers; $n = \pm 1, \pm 3, \dots, \pm(2m+1), \dots$ for $k = 1, 3, 5, \dots$, and $n = 0, \pm 2, \pm 4, \dots, \pm 2m, \dots$ for $k = 0, 2, 4, \dots$, $u_n^0 = u_0(nl)$.

If $\varphi(u)$ is an arbitrary smooth function, then the proof is carried out for any bounded monotone function $u_0(x)$. In the case when φ'' preserves its sign, we shall prove the convergence of the scheme also for any bounded $u_0(x)$, if $|u_0'(x)| \leqslant M_1$. Using Lemma 5 from ([3]), it is then easy to obtain convergence of the scheme also for any bounded measurable $u_0(x)$.

Writing the difference scheme (3) in the form

$$u_n^{k+1} = \left(\tfrac{1}{2} - \varphi' h/2l\right) u_{n+1}^k + \left(\tfrac{1}{2} + \varphi' h/2l\right) u_{n-1}^k$$

and assuming that $|\varphi'(u)|\, h/l < 1$ for $|u| \leqslant M$, we easily obtain that $|u_n^k| \leqslant M$ and that $u_{n+1}^k - u_{n-1}^k \leqslant 0$ ($\geqslant 0$), if $u_{n+1}^0 - u_{n-1}^0 \leqslant 0$ ($\geqslant 0$), $k = 1, 2, \dots$. In the case $\varphi'' \geqslant 0$ ($\leqslant 0$) it is easy to prove the inequality (see ([3]))

$$z_n^k \equiv (u_n^k - u_{n-2}^k)/2l \leqslant M_1 \ (\geqslant M_1). \tag{4}$$

We shall assume that

$$\int_{-\infty}^{x} |u_0(x) - u_-|\, dx < \infty$$

and that $u_0(x)$ tends monotonically to u_- as $x \to -\infty$. Then there exists

$$\bar{u}_n^k = \sum_{m=-\infty}^{n} (u_m^k - u_-)\, 2l$$

for every point $(t, x) = (kh, nl)$, and $\bar{u}_n^k$ are uniformly bounded for $x < x_0$, $0 \leq t \leq t_0$. Indeed, the existence of $\bar{u}_n^k$ successively for $k = 1, 2, \dots$ follows from the equality

$$\bar{u}_n^{k+1} = \frac{1}{2}(\bar{u}_{n+1}^k + \bar{u}_{n-1}^k) - h[\varphi(u_{n+1}^k) - \varphi(u_-)].$$

This equality can be written in the form

$$\bar{u}_n^{k+1} = (1/2 - \tilde{\varphi}' h/2l)\bar{u}_{n+1}^k + (1/2 + \tilde{\varphi}' h/2l)\bar{u}_{n-1}^k.$$

From the last expression for $\bar{u}_n^{k+1}$ it follows that

$$|\bar{u}_n^{k+1}| \leq \max_m |u_0(ml)|,$$

where $n-(k+1) \le m \le n+(k+1)$.

Let u_n^k be a solution of (3) with steps h and l in t and x, respectively, and let v_n^k be a solution of the same scheme with steps $h/2$ and $l/2$. For convenience we shall use half-integer indices for v_n^k: $n = 0, \pm 1, \pm 2, \ldots$ for $k = 0, 1, 2, \ldots$, and $n = \pm 1/2, \pm 1^1/2, \ldots$ for $k = 1/2, 1^1/2, \ldots$. We assume that $\varphi(u_-) = 0$. We have

$$v_n^{k+1} = \frac{1}{2}[v_{n+1/2}^{k+1/2} + v_{n-1/2}^{k+1/2}] - \varphi(v_{n+1/2}^{k+1/2}) - [\varphi(v_{n-1/2}^{k+1/2})]\,[h/2l].$$

For

$$\bar{v}_n^k = \sum_{m=-\infty}^{n} [v_m^k - u_-]\, l$$

we obtain

$$\begin{aligned}
\bar{v}_n^{k+1} &= \frac{1}{2}(\bar{v}_{n+1/2}^{k+1/2} + \bar{v}_{n-1/2}^{k+1/2}) - \varphi(v_{n+1/2}^{k+1/2})\, h/2 \\
&= \frac{1}{2}\left(\frac{1}{2}\bar{v}_{n+1}^k + \frac{1}{2}\bar{v}_{n-1}^k\right) + \frac{1}{2}\left(\frac{1}{2}\bar{v}_{n-1}^k + \frac{1}{2}\bar{v}_n^k\right) - \varphi(v_{n+1}^k)\, h/4 - \varphi(v_n^k)\, h/4 - \varphi(v_{n+1/2}^{k+1/2})\, h/2.
\end{aligned}$$

Let

$$W_n^k = \bar{u}_n^k - \bar{v}_n^k.$$

For W_n^k we obtain the equation

$$\begin{aligned}
W_n^{k+1} &= \frac{1}{2}(W_{n+1}^k + W_{n-1}^k) + \frac{1}{4}\bar{v}_{n+1}^k + \frac{1}{4}\bar{v}_{n-1}^k - \frac{1}{2}\bar{v}_n^k \\
&\quad - h\varphi(u_{n+1}^k) + h\varphi(v_{n+1}^k) - \frac{3}{4}h\varphi(v_{n+1}^k) + \frac{1}{4}h\varphi(v_n^k) + \frac{1}{2}h\varphi(v_{n+1/2}^{k+1/2}).
\end{aligned}$$

Transforming the terms on the right-hand side with the aid of the mean value theorem, we obtain

$$\begin{aligned}
W_n^{k+1} &= (1/2 - \tilde{\varphi}' h/2l) W_{n+1}^k + (1/2 + \tilde{\varphi}' h/2l) W_{n-1}^k \\
&\quad + (v_{n+1}^k - v_n^k)\, l\, [1/4 + \varphi_1 h/2l - \varphi_2 h/4l - \varphi_3 h/4l - \varphi_4 \varphi_5 h^2/4l^2],
\end{aligned}$$

where $\tilde{\varphi}'$ and φ_i $(i = 1, \ldots, 5)$ denote $\varphi'(u)$ for certain values of u not exceeding M in absolute value. Let h/l be so small that the expression in the last brackets is nonnegative and less than some number M_2. Suppose that $\varphi'' \geq 0$ (the other cases are considered analogously). Using (4), we obtain

$$W_n^{k+1} - [1/2 - \tilde{\varphi}' h/2l] W_{n+1}^k - [1/2 + \tilde{\varphi}' h/2l] W_{n-1}^k \leq h M_1 M_2 l^2 / h$$

and

$$|W_n^0| \leq M_3 l,$$

if $u_0(x)$ has bounded variation for $x \leq x_0$ (M_3 depends on x_0). For $nl < x_0 - (k+1)l$ it follows from this that

$$W_n^{k+1} \leq M_3 l + (k+1) h M_1 M_2 l^2 / h$$

or

$$W_n^k \equiv W(kh, nl) \leq C_1 l + C_2 l^2 / h$$

for $t \leq t_0$ and $x \leq x_0 - t_0 l/h$, where C_1 and C_2 are certain constants.

Consequently,

$$[\bar{u}_n^k - C_2\, 2l - \tfrac{1}{2} C_2 (2l)^2 / h] - [\bar{v}_n^k - C_1 l - \tfrac{1}{2} C_2 2l^2 / h] \leq 0,$$

and the function

$$Y_n^k \equiv \bar{u}_n^k - C_1\, 2l - \tfrac{1}{2} C_2 (2l)^2 / h,$$

which depends on the mesh steps l and h, increases monotonically when l and h tend to zero as $l_0/2^s$ and $h_0/2^s$ as $s \to +\infty$. Since Y_n^k are uniformly bounded for $kh = t \leq t_0$, $nl = x \leq x_0 - t_0 h/l$ for arbitrary $t_0 \geq 0$ and x_0, it follows that Y_n^k converge when h and l tend to zero. Hence the convergence of $\bar{u}_n^k$ follows as $h \to 0$ and $l \to 0$.

Extend u_{hl} for all x and $t \geq 0$, putting $u_{hl}(t, x) = u_n^k$ for $(k-1)h < t \leq kh$, $(n-2)l < x \leq nl$. From the convergence of $\bar{u}_n^k$ follows the weak convergence of u_{hl} on all straight lines $t = \text{const}$ containing mesh points. Writing (3) in the form

$$(u_n^{k+1} - u_n^k)/h + [\varphi(u_{n+1}^k) - \varphi(u_{n-1}^k)]/2l = (u_{n+1}^k - 2u_n^k + u_{n-1}^k)/2h,$$

it is not difficult

obtain that

$$\sum_{n=n_1}^{n_2} (u_n^k - u_n^{k-m})\, 2l \leqslant C_3 mh.$$

Using this, we obtain that the weak convergence of u_{hl} takes place on all straight lines $t = \text{const} \geqslant 0$. From the monotonicity with respect to n, or from (4) in the case $\varphi'' \neq 0$, compactness of u_{hl} follows in the sense of convergence almost everywhere. Therefore u_{hl} converge almost everywhere for $t \geqslant 0$, when $h, l \to 0$.

2. **The Rothe scheme for a quasilinear equation.** Here, as in § 1, we restrict ourselves to consideration of the simplest equation (1). The differential-difference Rothe scheme for solving the boundary-value problem

$$u\Big|_{t=0} = u_0(x), \qquad u\Big|_{x=0} = u_1(t) \tag{5}$$

in the domain $\{t \geqslant 0,\ x \geqslant 0\}$ for equation (1) is specified by the equations

$$du^{n+1}/dt + [\varphi(u^{n+1}) - \varphi(u^n)]/l = 0, \tag{6}$$

where $n = 0, 1, 2, \dots;$ $u^n(t) = u_l(t, nl);$ $u^0(t) = u_1(t);$ $u^n(0) = u_0(nl)$. Let $|u_0(x)| \leqslant M$, $|u_1(t)| \leqslant M$, and $\varphi'(u) > 0$ for $|u| \leqslant M + \varepsilon$, $\varepsilon > 0$. Considering successively $n = 1, 2, \dots$, it is easy to prove that $|u^n(t)| \leqslant M$ for all n.

We shall assume that $\varphi'' \geqslant 0$. We shall prove that in this case

$$z^n \equiv (u^n - u^{n-1})/l \leqslant M_1 \quad \text{for } n = 1, 2, \dots \tag{7}$$

Define z^n for $n = 0$. For this, define $u^{-1}(t)$ from the equation

$$\varphi(u^{-1}) = \varphi(u_1(t)) + l\, du_1/dt. \tag{8}$$

It is obvious that for sufficiently small l this equation has a solution with respect to u^{-1}, and $|u^{-1}| \leqslant M + \varepsilon/2$. It is easy to see that $z^0 = u_1'(t)/\varphi' \leqslant M_1$, if $u_1'(t) \leqslant M_2$ (M_1 and M_2 are certain constants). The function z^n satisfies the equation

$$dz^n/dt + \varphi'(u^{n-1})(z^n - z^{n-1})/l + \frac{1}{2}\widetilde{\varphi''}(z^n)^2 + \frac{1}{2}\widetilde{\widetilde{\varphi''}}(z^{n-1})^2 = 0. \tag{9}$$

It follows from this that $z^n \leqslant M_1$, if $z^n(0) \leqslant M_1$ and $z^0(t) \leqslant M_1$, since otherwise we would obtain a contradiction with equation (9) at a maximum point of $z^n(t)$ for $t \leqslant t_0$ and $ln \leqslant x_0$.

Let v^n be solutions of equations (6) with step $l/2$ ($n = 0, \frac{1}{2}, 1, 1\frac{1}{2}, ...$) and with the conditions $v^n(0) = u_0(nl)$, $v^0(t) = u_1(t)$. Introduce the notation:

$$\overline{u}^n = \sum_{m=0,1,2,...}^{n} u^m l, \qquad \overline{v}^n = \sum_{m=0,\frac{1}{2},1,...}^{n} v^m l/2.$$

The function $W^{n+1} = \overline{u}^{n+1} - \overline{v}^{n+1}$ satisfies the equation

$$dW^{n+1}/dt + \widetilde{\varphi'}[W^{n+1} - W^n]/l - \frac{1}{2}\widetilde{\varphi'}(v^{n+1} - v^{n+1/2}) - [\varphi(u^{-1}) - \varphi(v^{-1/2})].$$

Using (7) and (8), we find that

$$dW^{n+1}/dt + \widetilde{\varphi'}(W^{n+1} - W^n)/l \leqslant M_3 l,$$

where M_3 is a certain constant, $W^0(t) = 0$, $W^n(0) \leqslant M_4 l$ for $x \leqslant x_0$, if $|u_0'(x)| \leqslant M_5$. Hence it follows that $W^n(t) \leqslant M_6 l$, and $\overline{u}^n - 2M_6 l \leqslant \overline{v}^n - M_6 l$. Thus, the function $\overline{u}^n - 2M_6 l$ decreases monotonically as l tends to zero, like $l_0/2^s$ as $s \to \infty$. Hence follows the weak convergence of the functions u_l, and also, by virtue of (7), the convergence of u_l almost everywhere as $l \to 0$. Here we put $u_l(t, x) = u^n(t)$ for $(n-1)l < x \leqslant nl$.

In an analogous manner one can prove the convergence of the Rothe scheme for problem (1), (5) of the form

$$[u^{k+1} - u^k]/h + d\varphi(u^{k+1})/dx = 0, \qquad u^0(x) = u_0(x), \qquad u^k(0) = u_1(kh).$$

3. **A difference scheme for the filtration equation.** Consider the first boundary-value problem for the equation

$$\partial^2 u/\partial x^2 = \partial\Phi(u)/\partial t \tag{10}$$

in the rectangle $Q\{0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant 1\}$ with the conditions $u(0, x) = u_0(x)$, $u(t, 0) = u_1(t)$, $u(t, 1) = u_2(t)$, where $\Phi(u)$ is continuous for $u \geqslant 0$ and

$\Phi'(u) > 0$ for $u > 0$, and $u_0(x), u_1(t), u_2(t)$ are nonnegative functions. (The filtration equation (10) has been studied in detail in [4].) We shall consider the differential-difference equations for this problem in the form

$$\frac{d^2 u^k}{dx^2} = [\Phi(u^k) - \Phi(u^{k-1})]/h, \tag{11}$$

where

$$u^k(x) = u_h(kh, x), \quad k = 1, 2, \dots;$$

$$u^0(x) = u_0(x), \quad u^k(0) = u_1(kh), \quad u^k(1) = u_2(kh). \tag{12}$$

It is easy to show the existence of solutions u^k of equations (11), (12) and their uniform boundedness with respect to h. Denote by v^k the solution of the same difference scheme (11), (12) with step $h/2$. For convenience we shall use half-integer indices for v^k: $k = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots$.

Introduce the notation: $\bar{u}^k = \sum_{m=0}^{k} u^m h$, $\bar{v}^k = \sum_{m=0,\,1/2,\dots}^{k} v^m h/2$. We have $d^2\bar{u}^k/dx^2 = \Phi(u^k) - \Phi(u^0) + u_0'' h$, $d^2\bar{v}^k/dx^2 = \Phi(v^k) - \Phi(v^0) + u_0'' h/2$, $k = 1, 2, \dots$. For $W^k = \bar{u}^k - \bar{v}^k$ we obtain the equation

$$\frac{d^2W^k}{dx^2} = \tilde{\Phi}'\,(W^k - W^{k-1})/h + \frac{1}{2}\tilde{\Phi}'\,(v^{k-1/2} - v^k) + u_0'' h/2;$$

$$W^0 = 0, \quad |W^k(0)| \le M_1 h, \quad |W^k(1)| \le M_2 h \tag{13}$$

under the assumption that $u_1(t)$ and $u_2(t)$ have bounded variation. If the condition

$$v^k - v^{k-1/2} \le 0, \tag{14}$$

is fulfilled, then

$$\frac{d^2W^k}{dx^2} - \tilde{\Phi}'\,(W^k - W^{k-1})/h \ge u_0''(x)h/2 \ge M_3 h. \tag{15}$$

From (13) and (15) it follows that $W^k \le M_4 h$, $(\bar{u}^k - 2M_4 h) - (\bar{v}^k - 2M_4 h/2) \le 0$, i.e. $\bar{u}^k - 2M_4 h$ is a monotonically decreasing function of the step h. Hence, as in paragraphs 1 and 2, the weak convergence of u_h as $h \to 0$ follows. Using (14), we obtain convergence of u_h almost everywhere in Q. (We put $u_h(t, x) = u^k(x)$ for $(k-1)h < t \le kh$.)

The proof of convergence of the scheme (11), (12) is carried out analogously in the case when

$$v^k - v^{k-1/2} \ge 0. \tag{16}$$

It is easy to show that condition (14) is fulfilled if $u_0''(x) \le 0$, $u_1(t)$ and $u_2(t)$ are nonincreasing functions of t, while condition (16) is fulfilled if $u_0''(x) \ge 0$, and u_1 and u_2 are nondecreasing functions of t. For this it is necessary to write the equation for $z^k = u^k - u^{k-1}$ and use the maximum principle.

We note that the indicated method is applicable for proving the convergence of a wide class of difference and differential-difference schemes possessing the maximum principle.

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Note: Figure translations are in progress. See original paper for figures.