PHYSICS

1961

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Abstract

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PHYSICS

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ONE-DIMENSIONAL ADIABATIC FLOWS OF AN ULTRARELATIVISTIC GAS

(Presented by Academician N. N. Bogolyubov, March 4, 1961)

From the general equations of relativistic hydrodynamics one can arrive at two equations describing relativistic motions of a medium possessing point symmetry (^{1}):

$$\left(\frac{\partial u}{\partial s} + u\frac{\partial u}{\partial r}\right)\left(1+\frac{p}{\varepsilon}\right) + \frac{u}{\varepsilon}\frac{\partial p}{\partial s} + \frac{1+u^2}{\varepsilon}\frac{\partial p}{\partial r} = 0; \tag{1}$$

$$\frac{\partial \ln\varepsilon}{\partial p} + u\frac{\partial \ln\varepsilon}{\partial r} + \left(1+\frac{p}{\varepsilon}\right)\left(\frac{\partial u}{\partial r} + \frac{u}{1+u^2}\frac{\partial u}{\partial s} + \frac{Nu}{r}\right) = 0; \tag{2}$$

here u is the 4-velocity; $u = a/c\sqrt{1-a^2/c^2}$, where a is the ordinary velocity; p is the pressure; $\varepsilon = \rho c^2$ is the energy density (ρ is the density of the medium); $ds = c\,dt\sqrt{1-a^2/c^2}$ is the interval; $N = 0;\ 1;\ 2$, respectively, for plane, cylindrical, and spherical waves.

In studying adiabatic flows of a medium in the relativistic case, to these equations one must add the adiabaticity equation:

$$\frac{\partial \sigma}{\partial t} + u\frac{\partial \sigma}{\partial r} = 0, \tag{3}$$

where s is the entropy.

In the case of the equation of state

$$pv^k = A(\sigma), \tag{4}$$

where v is the specific volume, we shall have

$$\varepsilon = c^2\beta(\sigma)p^{1/k} + \frac{p}{k-1}, \tag{5}$$

where $\beta(\sigma) = A^{-1/k}(\sigma)$, and therefore $\varepsilon v = c^2 + pv/(k-1)$. Taking into account this relation between ε, $A(\sigma)$, and p, equation (3) can be written in the form

$$\frac{\varepsilon + p}{kp}\left(\frac{\partial p}{\partial s} + u\frac{\partial p}{\partial r}\right) = \frac{\partial \varepsilon}{\partial s} + u\frac{\partial \varepsilon}{\partial r}. \tag{6}$$

The system of equations (1), (2), and (6) completely describes adiabatic flows of a relativistic gas.

Let us note that in this case the number of particles per unit volume $n \sim 1/v$, and the law of conservation of the number of particles holds:

$$-\left(\frac{\partial \ln v}{\partial s} + u\frac{\partial \ln v}{\partial r}\right) + \frac{\partial u}{\partial r} + \frac{u}{1+u^2}\frac{\partial u}{\partial s} + \frac{Nu}{r} = 0, \tag{7}$$

where $d\ln n = -d\ln v = d\ln\rho/(1 + p/\rho c^2)$. In the case of an ultrarelativistic gas

$$p = (k-1)\varepsilon, \tag{8}$$

where $k = 4/3$; however, for the sake of generality of the formal solution we write the relation between p and ε precisely in the form (8). In this case equation (6) is satisfied identically. Equation (4) still remains in force, and the function $\beta(\sigma) = 0$. Thus, in order to study the motions of a medium in the ultrarelativistic case it is necessary to solve only the two equations (1) and (2).

If it is required to find the dependence of the specific volume on s, r, then one should solve the equation

$$\frac{d(p\tilde{v}^k)}{ds} = \frac{\partial(pv^k)}{\partial s} + u\frac{\partial(pv^k)}{\partial r} = 0 \tag{9}$$

for an already known function $p = p(s, r)$.

In the ultrarelativistic case the law $n \neq 1/v$, and the conservation law for the number of particles is no longer applicable, since pair creation and annihilation occur.

Introducing $p^* = \ln p^{(k-1)/k}$, $\operatorname{sh}\omega = u$, and $ds = d\tau/\operatorname{ch}\theta$, where $\tau = ct$, we write the system (1) and (2) in the form

$$\frac{\partial \omega}{\partial \tau} + \operatorname{th}\omega\frac{\partial \omega}{\partial r} + \frac{\partial p^*}{\partial r} + \operatorname{th}\omega\frac{\partial p^*}{\partial \tau} = 0, \tag{10}$$

$$\frac{\partial p^*}{\partial \tau} + \operatorname{th}\omega\frac{\partial p^*}{\partial r} + (k-1)\left[\frac{\partial \omega}{\partial r} + \operatorname{th}\omega\frac{\partial \omega}{\partial \tau} + \frac{N}{r}\operatorname{th}\omega\right] = 0. \tag{11}$$

This system is easily solved by the method of characteristics. Along the lines

$$\frac{dr}{dt} = \frac{\operatorname{th}\omega \pm \sqrt{k-1}}{1 \pm \sqrt{k-1}\operatorname{th}\omega}$$

the condition

$$dp^* \pm \sqrt{k-1}\,d\omega = -(k-1) \times \frac{\operatorname{th}\omega\,(N/r)\,d\tau}{1 \pm \sqrt{k-1}\operatorname{th}\omega}$$

is satisfied. Passing to the ordinary velocity $a/c = \operatorname{th}\omega$ and pressure, these relations may be written in the form

$$\frac{dr}{d\tau} = \frac{a/c \pm \sqrt{k-1}}{1 \pm \sqrt{k-1}\,a/c}; \tag{12}$$

$$d\left[\ln p\left(\frac{1+a/c}{1-a/c}\right)^{\pm k/2\sqrt{k-1}}\right] = -\frac{kN\,d\tau}{r}\,\frac{a/c}{1 \pm \sqrt{k-1}\,a/c}. \tag{13}$$

In the case $N = 0$, for one-dimensional motions of a gas, the system of equations (10) and (11) admits exact solutions.

We shall seek them in two ways: in ordinary coordinates and in Lagrangian coordinates. In the first case, assuming that $\partial(p^*, \omega)/\partial(\tau, x) \neq 0$, where $x = r$, we interchange dependent and independent variables in equations (10), (11), which gives

$$\operatorname{th}\omega\left(\frac{\partial x}{\partial \omega} + \frac{\partial \tau}{\partial p^*}\right) = \frac{\partial \tau}{\partial \omega} + \frac{\partial x}{\partial p^*}; \tag{14}$$

$$\frac{\partial x}{\partial \omega} + (k-1)\frac{\partial \tau}{\partial p^*} = \operatorname{th}\omega\left(\frac{\partial \tau}{\partial \omega} + (k-1)\frac{\partial x}{\partial p^*}\right). \tag{15}$$

Assuming that ([2,3])

$$x = e^{-p^*}\left(\frac{\partial \psi}{\partial p^*}\operatorname{sh}\omega - \frac{\partial \psi}{\partial \omega}\operatorname{ch}\omega\right); \qquad \tau = e^{-p^*}\left(\frac{\partial \psi}{\partial p^*}\operatorname{ch}\omega - \frac{\partial \varphi}{\partial \omega}\operatorname{sh}\omega\right) \tag{16}$$

where $\psi = \psi(p^*; \theta)$, we identically satisfy equation (14), while equation (15) takes the form

$$\frac{\partial^2 \psi}{\partial \omega^2} = (k-1)\frac{\partial^2 \psi}{\partial p^{*2}} + (2-k)\frac{\partial \psi}{\partial p^*}. \tag{17}$$

Reducing the equation to characteristics by introducing

$$\alpha_1 = \frac{2-k}{4(k-1)}\left[\sqrt{k-1}\,\omega + p^*\right],$$

$$\alpha_2 = \frac{2-k}{4(k-1)}\left[\sqrt{k-1}\,\omega - p^*\right]$$

and putting $\psi = e^{\alpha_2 - \alpha_1}\varphi$, we arrive at the Riemann equation

$$\frac{\partial^2\varphi}{\partial\alpha_1\partial\alpha_2} = -\varphi. \qquad (18)$$

For any initial and boundary conditions (satisfying the equation) one can find $\varphi = \varphi(a_1, a_2)$; then it is already easy to determine $\psi = \psi(p^*, \omega)$, $x = x(p^*, \omega)$, $\tau = \tau(p^*, \omega)$. Equation (9), referred to the independent variables (p^*, ω), takes the form

$$-\frac{\partial \ln v^{k-1}}{\partial p^*} + \frac{\partial \ln v^{k-1}}{\partial \omega}\,\frac{\partial(\partial\psi/\partial p^* - \psi)/\partial\omega}{\partial^2\psi/\partial\omega^2 - \partial\psi/\partial p^*} = 1. \qquad (19)$$

This equation serves to determine v when $\psi = \psi(p^*, \omega)$ is already known. In the case when $\partial(p^*, \omega)/\partial(\tau, x) = 0$, we are, as is known, dealing with a special solution, which has the form

$$x = \frac{\operatorname{th}\omega \pm \sqrt{k-1}}{1 \pm \sqrt{k-1}\,\operatorname{th}\omega}\,\tau + F(\omega); \qquad p^* = \pm\sqrt{k-1}\,\omega + \text{const} \qquad (20)$$

or

$$x = \frac{a/c \pm \sqrt{k-1}}{1 + \sqrt{k-1}\,a/c}\,\tau + F\left(\frac{a}{c}\right); \qquad \left(\frac{1 + a/c}{1 - a/c}\right) p^{\pm 2\sqrt{k-1}/k} = \text{const}. \qquad (21)$$

In this case, to determine v we have the solution that follows directly from equation (9), transformed to the independent variables (ω, v):

$$\tau = (1 + \operatorname{th}\omega)^{\frac{-2(2-k)}{\pm\sqrt{k-1}}(1\mp\sqrt{k-1})}(1 - \operatorname{th}\omega)^{\frac{2(2-k)}{\pm\sqrt{k-1}}(1\pm\sqrt{k-1})}(1 \pm \sqrt{k-1}\,\operatorname{th}\omega)\times$$

$$\times\left[\Phi\left(\ln v^{\pm\sqrt{k-1}} - \omega\right) \mp \frac{1}{\sqrt{k-1}}\int dF(\omega)\,\operatorname{ch}^2\omega\,(1 + \operatorname{th}\omega)^{\frac{2(2-k)}{\pm\sqrt{k-1}}(1\mp\sqrt{k-1})}\times\right.$$

$$\times(1-\operatorname{th}\omega)^{\frac{-2(2-k)}{\pm\sqrt{k-1}}(1\pm\sqrt{k-1})}\Big]. \tag{22}$$

Let us now consider the solution in Lagrangian coordinates (τ, r_0). Since $(\partial r/\partial\tau)_{r_0} = \partial r/\partial\tau = \operatorname{th}\omega$, equations (10), (11) take the form:

$$\frac{\partial\omega}{\partial\tau} + \frac{1}{\partial r/\partial r_0}\frac{\partial p^*}{\partial r_0}\frac{1}{\operatorname{ch}^2\omega} + \operatorname{th}\omega\frac{\partial p^*}{\partial\tau} = 0; \tag{23}$$

$$\frac{1}{k-1}\frac{\partial p^*}{\partial\tau} + \frac{1}{\partial r/\partial r_0}\frac{\partial\omega}{\partial r_0}\frac{1}{\operatorname{ch}^2\omega} + \operatorname{th}\omega\frac{\partial\omega}{\partial r} + \operatorname{th}\omega\frac{N}{r} = 0. \tag{24}$$

The last equation is integrated:

$$\frac{1}{F(r_0)}\frac{\partial r}{\partial r_0} = \frac{p^{-1/k}r^{-N}}{\operatorname{ch}\omega} = e^{-p^*/(k-1)}r^{-N}\frac{1}{\operatorname{ch}\omega}.$$

Eliminating, with the aid of this equation, the derivative $\partial r/\partial r_0$ from (23) and (24), we arrive at the result:

$$\operatorname{ch}\omega\frac{\partial\omega}{\partial r} + \frac{\partial p^*}{\partial m}e^{p^*/(k-1)}r^N + \operatorname{sh}\omega\frac{\partial p^*}{\partial\tau} = 0, \tag{25}$$

where $dm = F(r_0)\,dr_0$;

$$\frac{\operatorname{ch}\omega}{k-1}\frac{\partial p^*}{\partial\tau} + \operatorname{sh}\omega\frac{\partial\omega}{\partial\tau} + \frac{\partial\omega}{\partial m}r^N e^{p^*/(k-1)} + \operatorname{sh}\omega\frac{N}{2} = 0. \tag{26}$$

These equations can be solved by the method of characteristics, which in Lagrangian coordinates have the form: along the lines

$$\frac{d\tau}{dm} = r^{-N}e^{-p^*/(k-1)}\left(\operatorname{sh}\omega \mp \frac{\operatorname{ch}\omega}{\sqrt{k-1}}\right)$$

the relations

$$dp^* = \pm\sqrt{k-1}\left(\operatorname{ch}\omega + Nr^{-(N+1)}e^{-p^*/(k-1)}\operatorname{sh}\omega\,dm\right)$$

are satisfied.

Let us now consider exact solutions for $N = 0$. We take as dependent and independent variables of the system (25) and (26), assuming that $\partial(p^*,\omega)/\partial(\tau,m) \neq 0$. As a result we shall have

$$\frac{\partial\tau}{\partial\omega} = e^{-p^*/(k-1)}\left(\operatorname{sh}\omega\frac{\partial m}{\partial\omega} - \operatorname{ch}\omega\frac{\partial m}{\partial p^*}\right); \tag{27}$$

$$\frac{\partial \tau}{\partial p^*} = e^{-p^*/(k-1)} \left(\operatorname{sh}\omega \, \frac{\partial m}{\partial p^*} - \operatorname{ch}\omega \, \frac{\partial m}{\partial \omega} \right). \tag{28}$$

Eliminating τ by differentiation, we arrive at the equation

$$\frac{\partial^2 m}{\partial \omega^2} = (k-1)\frac{\partial^2 m}{\partial p^{*2}} - (2-k)\frac{\partial m}{\partial p^*}. \tag{29}$$

This equation is very similar to equation (17) and differs from it only in the sign of the first derivative. Referring the equation to the characteristics

$$\alpha_1 = -\frac{2-k}{4(k-1)} \left(\sqrt{k-1}\,\omega + p^* \right), \quad \alpha_2 = -\frac{2-k}{4(k-1)} \left(\sqrt{k-1}\,\omega - p^* \right)$$

and putting $m = e^{\alpha_1 - \alpha_2}\varphi$, we arrive at the Riemann equation

$$\frac{\partial^2 \varphi}{\partial \alpha_1 \, \partial \alpha_2} = -\varphi. \tag{30}$$

Determining $\varphi = \varphi(\alpha_1, \alpha_2)$ for any specific conditions of the problem, it is easy to determine $m = m(p^*, \omega)$, after which $\tau = \tau(p^*, \omega)$ can be determined from equation (27) or (28). Since in Lagrangian coordinates $\partial(pv^k)/\partial s = 0$, i.e. $pv^k = f^k(m)$, and $m = m(\sigma)$, from this is determined

$$v = p^{-1/k} f[m(p, \omega)]. \tag{31}$$

If $\partial(p^*, \omega)/\partial(\tau, m) = 0$, then we have a special solution which, in Lagrangian coordinates, has the form

$$m = \frac{e^{p^*/(k-1)}\tau}{\operatorname{sh}\omega \pm \operatorname{ch}\omega/\sqrt{k-1}} + F(p^*), \quad p^* = \pm\sqrt{k-1}\,\omega + \text{const}. \tag{32}$$

In the ordinary variables of velocity and pressure the special solution has the form

$$m = \frac{p^{1/k}\sqrt{1 - a^2/c^2}\,\tau}{a/c \pm 1/\sqrt{k-1}} + F\left(\frac{a}{c}\right) = A\,\frac{\left(\dfrac{1 + a/c}{1 - a/c}\right)^{\pm 1/2\sqrt{k-1}} \sqrt{1 - a^2/c^2}\,\tau}{a/c \pm 1/\sqrt{k-1}} \tag{33}$$

where $A = \text{const}$. As before, $v = p^{1/k} f(m)$.

Thus, we are able to solve arbitrary problems of propagation of both simple and reflected waves, including shock waves.

The solution of an analogous problem in Eulerian coordinates was given earlier by I. M. Khalatnikov (^{2}). However, in his solution assumptions were made that restrict the generality of the problem. His resulting equation (for $k = 4/3$) had the form $3\partial^2\psi/\partial\omega^2 = \partial^2\psi/\partial y^2 + 2\partial\psi/\partial y$, $y = \ln T$, where T is the temperature. In fact, I. M. Khalatnikov assumed that an ultrarelativistic gas is completely similar to a photon gas, for which the thermodynamic potential $\mu = 0$; this applies when considering the problem of multiple particle production, which Khalatnikov was solving. However, for electron and neutron ultrarelativistic gases the number of "seed" particles may be conserved, and in this case $\mu \neq 0$. In this case Khalatnikov' s equation proves suitable only for isentropic motions, since for them $T = \text{const} \cdot p^{1/4}$. In the case of adiabatic motions, when shock waves can be considered, $T = f(\sigma)p^{1/4}$, and Khalatnikov' s solution ceases to "work." In the case of an ultrarelativistic gas with $\mu = 0$, also for adiabatic motions $T = \text{const} \cdot p^{1/4}$, and, consequently, Khalatnikov' s solution is suitable at $\mu = 0$ for motions with shock waves.

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Received
1 III 1961

REFERENCES

1. K. P. Stanyukovich, *Proceedings of the III Conference on Problems of Cosmogony*, Publishing House of the Academy of Sciences of the USSR, 1954, p. 279.
2. I. M. Khalatnikov, ZhETF, **27**, no. 5 (11) (1954).
3. F. A. Baum, S. A. Kaplan, K. P. Stanyukovich, *Introduction to Cosmic Gas Dynamics*, part 3, ch. 1, § 3, 1958.

Note: Figure translations are in progress. See original paper for figures.

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