MECHANICS

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1961

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE MOTION OF A GYROSCOPE IN A GIMBAL SUSPENSION IN THE PRESENCE OF A MOMENT ABOUT THE AXIS OF PROPER ROTATION

(Presented by Academician A. Yu. Ishlinskii, January 6, 1961)

1°. The moment of resistance to the proper rotation of a gyroscope considerably exceeds the friction moments in the axes of the inner and outer rings; therefore the integral of proper rotation is no longer applicable. In order that the proper angular momentum of the gyroscope not change, an artificial driving moment is created, balancing the resisting moment. If the suspension rings are not perpendicular, the driving moment has a damping effect on the nutational oscillations of the gyroscope, leading to a systematic drift ([1–3]).

2°. The equations of motion of the mechanical system can be obtained from d' Alembert' s generalized principle

$$\sum_{i=1}^{3}\left[\frac{d}{dt}\left(\frac{dT}{d\dot{q}_i}\right)-\frac{dT}{dq_i}-Q_i\right]\delta q_i=0.$$

If we choose the coordinate systems $O\xi\eta\zeta$, $Ox_2y_2z_2$, $Ox_1y_1z_1$, and $Oxyz$, associated respectively with the inertial space, the outer ring, the inner ring, and the rotor, then the position of the system can be determined uniquely by three angles α, β, γ (Fig. 1). It is convenient to take these angles as the generalized coordinates of the system.

Fig. 1

Let us suppose that the coordinate axes are the principal axes of inertia of the outer ring, the inner ring, and the rotor, and denote the moment of inertia of the outer ring about its axis of rotation by A_2, the moments of inertia of the inner ring about the axes x_1, y_1, z_1, respectively, by A_1, B_1, C_1, and the equatorial

Fig. 2. 1—resistance torque, 2—driving torque

Figure 2: Fig. 2. 1—resistance torque, 2—driving torque

and polar moments of inertia of the rotor by A and C. The doubled kinetic energy of the system is

$$2T = \left[A_2 + (A + A_1)\cos^2\beta + C_1\sin^2\beta\right]\dot{\alpha}^2 + (A + B_1)\dot{\beta}^2 + C(\dot{\gamma} + \dot{\alpha}\sin\beta)^2.$$

If the gyroscope is astatic and there is no friction in the suspension axes, then the generalized forces with respect to the coordinates α and β are equal to zero. Consequently, the work of the external forces on a possible displacement $\delta\alpha, \delta\beta, \delta\gamma$ is $W = Q_\gamma \delta\gamma$, where Q_γ represents the sum of the resisting moment and the driving moment.

The dependences of the driving moment and of the resisting moment on the speed are represented graphically in Fig. 2. At the point of intersection of the two curves, the mo—

the resistance torque is equal to the driving torque; therefore, this point corresponds to the established rotational speed of the rotor ω_0.

Knowing the analytic expressions for these torques, one can establish the dependence of Q_γ on $s = \dot{\gamma} - \omega_0$; obviously, for $s = 0$, $Q_\gamma = 0$. Expanding $Q(s)$ in a series,

$$Q_\gamma(s) = \left.\frac{\partial Q_\gamma}{\partial s}\right|_{s=0} s + \frac{1}{2}\left.\frac{\partial^2 Q_\gamma}{\partial s^2}\right|_{s=0} s^2 + \cdots.$$

Fig. 2. 1—resistance torque, 2—driving torque

In view of the fact that below we shall investigate the oscillations of the gyroscope near the established motion, we retain in the expansion the linear term * . It is possible, of course, to include the function $Q_\gamma(s)$ in the investigation, but this only makes the calculations more complicated without adding anything new to the mechanics of the motion.

We are now in a position to write the equations of motion of the gyroscope

$$[A_2 + (A + A_1)\cos^2\beta + (C + C_1)\sin^2\beta]\ddot{\alpha}$$

$$-2(A + A_1 - C - C_1)\dot{\alpha}\dot{\beta}\cos\beta\sin\beta +$$

$$+H\dot{\beta}\cos\beta + C\left(\frac{ds}{dt}\sin\beta + s\dot{\beta}\cos\beta\right) = 0,$$

$$(A + B_1)\ddot{\beta} + (A + A_1 - C - C_1)\dot{\alpha}^2 \sin\beta \cos\beta - H\dot{\alpha}\cos\beta - Cs\dot{\alpha}\cos\beta = 0,$$

$$C\frac{d}{dt}(s + \dot{\alpha}\sin\beta) + \lambda s = 0,$$

where

$$H = C\omega_0, \qquad \lambda = -\left.\frac{\partial Q_\gamma}{\partial s}\right|_{s=0}.$$

The equations of motion have the first integral

$$[A_2 + (A + A_1)\cos^2\beta + (C + C_1)\sin\beta]\dot{\alpha} + (H + Cs)\sin\beta = k,$$

expressing the constancy of the projection of the kinetic moment of the system on the axis of the outer ring.

3°. The equations of motion admit the particular solution

$$\dot{\alpha} = 0, \qquad \dot{\beta} = 0, \qquad \alpha = \alpha_0, \qquad \beta = \beta_0, \qquad s = 0,$$

corresponding to the established rotation of the rotor with angular velocity ω_0 about its own axis, fixed in inertial space.

In order to investigate the stability of this motion, consider the perturbed motion

$$\alpha = \alpha_0 + \xi, \qquad \beta = \beta_0 + \eta, \qquad s = \zeta.$$

The positive definite function of the velocities

$$V(\xi, \eta, \dot{\eta}, \zeta) = [A_2 + (A + A_1)\cos^2(\beta_0 + \eta) + C_1\sin^2(\beta_0 + \eta)]\dot{\xi}^2 +$$

$$+ (A + B_1)\dot{\eta}^2 + C\big[\zeta + \dot{\xi}\sin(\beta_0 + \eta)\big]^2,$$

which represents twice the kinetic energy of the perturbed motion after subtracting $2C\omega_0^2$, has, by virtue of the equations of the perturbed motion, a constantly negative time derivative

$$\frac{dV}{dt} = -2\lambda\zeta^2,$$

* The problem of the spin-up of a gyroscope within the framework of precessional theory was solved by A. Yu. Ishlinskii in work (4).

if $\lambda > 0$. Consequently, according to Lyapunov' s theorem, the unperturbed motion is asymptotically stable with respect to the velocities ξ, η, and ζ.

4°. It is known that the nutational oscillations of a free gyroscope lead to a systematic drift of the gyroscope. However, in the presence of a rotational moment, the established motion is asymptotically stable with respect to $\dot{\alpha}$; therefore one should expect that, under the influence of nutational oscillations, the gyroscope will deviate through a certain angle. To determine this deviation, let us solve the equations of motion under the following initial conditions:

$$\alpha = \alpha_0, \qquad \dot{\alpha} = \Omega, \qquad \beta = \beta_0, \qquad \dot{\beta} = 0, \qquad s = 0.$$

We shall seek the solution of the equations in the form of series in powers of Ω

$$\alpha = \alpha_0 + \alpha_1\Omega + \alpha_2\Omega^2 + \dots, \qquad \beta = \beta_0 + \beta_1\Omega + \beta_2\Omega^2 + \dots,$$
$$s = s_1\Omega + s_2\Omega^2 + \dots .$$

The zero approximation coincides with the initial conditions. The equations of the first approximation,

$$[A_2 + (A + A_1)\cos^2\beta_0 + (C + C_1)\sin^2\beta_0]\ddot{\alpha}_1 + H\cos\beta_0\dot{\beta}_1 + C\sin\beta_0\dot{s}_1 = 0,$$
$$(A + B_1)\ddot{\beta}_1 - H\cos\beta_0\dot{\alpha}_1 = 0,$$
$$T(\dot{s}_1 + \dot{\alpha}_1\sin\beta_0) + s_1 = 0,$$

where $T = C/\lambda$ is the rotor time constant, are integrated under the initial conditions

$$\dot{\alpha}_1 = 1, \qquad \beta_1 = 0, \qquad \dot{\beta}_1 = 0, \qquad s_1 = 0.$$

The characteristic equation of the system has two zero roots, one real root, and two complex conjugate roots with negative real parts. It is not possible to find an analytic expression for the roots in terms of the coefficients of the characteristic equation. However, they can be found approximately for small values of $\sin\beta_0$. In this case we find the solutions of the equations with accuracy up to $\sin^3\beta_0$. These solutions have the form

$$\dot{\alpha}_1 = e^{-nt}\cos\omega t, \qquad \beta_1 = \frac{H\cos\beta_0}{(A + B_1)\omega^2}e^{-nt}(1 - \cos\omega t),$$

$$s_1 = -\sin\beta_0 \left[\frac{k^2}{1+k^2} e^{-nt} \cos\omega t - \frac{k}{1+k^2} e^{-nt} \sin\omega t\right].$$

Here the damping n and the oscillation frequency ω are determined by the formulas

$$n = \frac{k^2}{1+k^2} \frac{C\sin^2\beta_0}{2I_0 T}, \qquad \omega = \frac{H\cos\beta_0}{\sqrt{(A+B_1)I_0'}},$$

where

$$I_0 = A_2 + (A+A_1)\cos^2\beta_0 + C_1\sin^2\beta_0,$$
$$I_0' = A_2 + (A+A_1)\cos^2\beta_0 + \left(C_1 + \frac{C}{1+k^2}\right)\sin^2\beta_0.$$

The coefficient k is the ratio of the rotor time constant to the period of the nutational oscillations of a free gyroscope whose proper angular momentum is $C\omega_0$, i.e.

$$k = T\nu_0,$$

where ν_0 is determined by the formula

$$\nu_0 = \frac{H\cos\beta_0}{\sqrt{(A+B_1)I_0}}.$$

The equations of the second approximation

$$\begin{gathered}
\left[A_2 + (A+A_1)\cos^2\beta_0 + (C+C_1)\sin^2\beta_0\right]\ddot{\alpha}_2 + H\cos\beta_0\dot{\beta}_2 + C\sin\beta_0 s_2 \\
= 2(A+A_1-C-C_1)\cos\beta_0\sin\beta_0(\dot{\alpha}_1\dot{\beta}_1 + \ddot{\alpha}_1\beta_1) \\
+H\dot{\beta}_1\beta_1\sin\beta_0 - C\frac{d}{dt}(\dot{s}_1\beta_1)\cos\beta_0, \\
(A+B_1)\ddot{\beta}_2 - H\cos\beta_0\dot{\alpha}_2 = -(A+A_1-C-C_1)\dot{\alpha}^2\cos\beta_0\sin\beta_0 \\
-H\dot{\alpha}_1\dot{\beta}_1\cos\beta_0 + Cs_1\dot{\alpha}_1\cos\beta_0, \\
T(\dot{s}_2 + \ddot{\alpha}_2\sin\beta_0) + s_2 = -T\cos\beta_0\frac{d}{dt}(\dot{\alpha}_1\beta_1)
\end{gathered}$$

are integrated after substituting the results obtained in the first approximation, under the initial conditions

$$\dot{\alpha}_2 = 0, \qquad \beta_2 = 0, \qquad \dot{\beta}_2 = 0, \qquad s_2 = 0.$$

The aperiodic component of the angular velocity of interest to us is determined, up to terms of order $\sin^3 \beta_0$, by the formula

$$\bar{\dot{\alpha}}_2 = -\frac{\sin \beta_0}{2H \cos^2 \beta_0} \left(A_2 + C_1 + \frac{C}{1+k^2} \right) e^{-2nt}.$$

From this we can determine the limiting angle of deflection of the figure axis of the gyroscope under the influence of nutations:

$$\alpha(t) \xrightarrow[t\to\infty]{} \frac{\sin \beta_0}{2H \cos^2 \beta_0} \left(A_2 + C_1 + \frac{C}{1+k^2} \right) \frac{\Omega^2}{2n}.$$

5°. A very interesting case arises if λ is so large that the rotor time constant may be set equal to zero. From the standpoint of mechanics this case may be regarded as the presence of a servocoupling expressed by condition (5), $s \equiv 0$. In this case $n \to 0$; then nutational oscillations with frequency

$$\nu = \frac{H \cos \beta_0}{\sqrt{(A+B_1)\left[A_2 + (A+A_1)\cos^2 \beta_0 + (C+C_1)\sin^2 \beta_0\right]}}$$

will cause a systematic drift of the gyroscope with velocity

$$\bar{\dot{\alpha}} = -\frac{\sin \beta_0 \Omega^2}{2H \cos^2 \beta_0}(A_2 + C_1 + C).$$

6°. The magnitude of the damping coefficient becomes zero for $T = 0$ and $T = \infty$. It is easy to verify that the maximum value of the damping coefficient corresponds to the value $T = 1/\nu_0$, i.e., when the rotor time constant is equal to the period of nutation of the free gyroscope.

Let us take the numerical values $\beta_0 = 30°$, $I_0 = 8.75$ g·cm·s^2, $A+B_1 = 5$ g·cm·s^2, $\omega_0 = 1500$ s^{-1}; from these we find the frequency of nutational oscillations of the free gyroscope $\nu_0 = 975$ s^{-1}. Let $A_2 + C_1 = 5$ g · cm · s^2, $\Omega = 1$ s^{-1}; then for the values $k = 1$; 0.1 and 10 we find that the limiting values of α are respectively 0.65; 3.25, and 6.5 arc seconds.

It is natural that if the small oscillations of the gyroscope are forced, then drift of the gyroscope cannot be avoided ([2]). However, damping limits the drift velocity at resonance.

Received
5 I 1961

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Note: Figure translations are in progress. See original paper for figures.