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**Abstract**

**Full Text**

## MATHEMATICS

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# ONE GENERALIZATION OF THE MULTIDIMENSIONAL BOCHNER THEOREM

*(Presented by Academician S. L. Sobolev, 16 IX 1960)*

In this note the  $n$ -dimensional Bochner theorem on positive definite (p.d.) functions is generalized to the case when the role of  $e^{i\lambda_j x_j}$  is played by eigenfunctions of differential (and more general) operators. For  $n = 1$  the generalization was obtained by M. G. Krein <sup>(1)</sup> and by the author <sup>(2)</sup> (in the case of generalized kernels—by Maurin <sup>(3)</sup>); for  $n > 1$  substantial difficulties arise, connected with the necessity of extending Hermitian commuting operators to self-adjoint commuting ones. The considerations of this article correspond to the case when the closures of the operators are already self-adjoint; they are based on the theory of expansions in generalized eigenvectors (for the literature see <sup>(4)</sup>). The note is connected with the work of A. G. Kostyuchenko and B. S. Mityagin <sup>(5)</sup>, in which similar results were obtained for kernels generating a commutative ring. We do not assume a connection with rings, thanks to which the class of differential operators is substantially enlarged (for example, operators of order higher than the second appear)\*.

1°. Let  $H_0$  be a complete Hilbert space with scalar product  $(f, g)_0$ ; let  $H_+$  be a linear set from  $H_0$ , dense in it and itself a complete Hilbert space with respect to another scalar product  $(u, v)_+$ , such that  $\|u\|_0 \leq \|u\|_+$  ( $u \in H_+$ ). The bilinear form  $B(f, u) = (f, u)_0$  is continuous in  $f \in H_0$  and  $u \in H_+$ ; therefore it can be written in the form  $(f, u)_0 = (If, u)_+$ , where  $I$  is an operator continuously acting from  $H_0$  into  $H_+$ . Introduce in  $H_0$  a new scalar product  $(f, g)_- = (If, g)_0$  and, carrying out the completion, obtain the Hilbert space  $H_-$ . Thus,

$$H_- \supset H_0 \supset H_+, \quad (1)$$

where each space of this chain is dense in the one standing to its left, and  $\|u\|_- \leq \|u\|_0 \leq \|u\|_+$ .

It is not difficult to show that a bilinear form  $(\alpha, u)_0$  ( $\alpha \in H_-$ ,  $u \in H_+$ ) is defined, becoming the scalar product in  $H_0$  if  $\alpha \in H_0$ . Every linear continuous functional  $l(u)$  on  $H_+$  can be written in the form  $l(u) = (\alpha, u)_0$ , where  $\alpha$  is some element of  $H_-$ . Thus,  $H_-$  may be interpreted as the space of linear functionals on  $H_+$  (it is clear that it is isometric to  $H_+$ ). Elements of the spaces  $H_+$ ,  $H_0$ , and  $H_-$ —spaces with positive, zero, and negative norms—we shall denote respectively

by  $u, v, \dots; f, g, \dots$  and  $\alpha, \beta, \dots$ . The space  $H_-$  may be regarded as the space of “generalized vectors” over the basic space  $H_+$  of “smooth vectors.”

Let the space  $H_{++}$  be in relation to  $H_+$  in the same position as  $H_+$  is in relation to  $H_0$ . Then, from  $H_{++}$  and  $H_0$ , one can construct the negative space  $H_{--}$ . It is easy to see that

$$H_{--} \supset H_- \supset H_0 \supset H_+ \supset H_{++}$$

with the same density relations and inequalities as in (1). We call the space  $H_+$  nuclear with respect to  $H_0$  if  $I$ , considered as an operator in  $H_0$ , has finite trace.

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\* The main constructions of the article were reported by the author at a seminar at the Institute of Mathematics of the Academy of Sciences of the USSR in the summer of 1959 (see also <sup>(4)</sup>). What is new is Theorem 3, instead of which a cruder result had previously been used, leading to estimates more restrictive than (6).

2°. Tensor-multiplying the spaces of the chain (1) by themselves, we obtain the chain  $H_- \times H_- \supset H_0 \times H_0 \supset H_+ \times H_+$ . It can be shown that the space  $H_- \times H_-$  will be negative with respect to the zero space  $H_0 \times H_0$  and positive  $H_+ \times H_+$ . The elements of the spaces  $H_+ \times H_+$ ,  $H_0 \times H_0$ , and  $H_- \times H_-$ , by analogy with the case of spaces of functions, will be called respectively smooth, ordinary, and generalized kernels.

In all that follows we shall assume that in the space  $H_0$  an involution  $f \rightarrow \bar{f}$  has been introduced, which is also an involution for  $H_+$ . Now the notions of Hermitian and positive definite kernels are introduced in the following way: to each generalized kernel  $K$  there corresponds the bilinear form  $B_K(u, v) = (K, \bar{v}u)_{H_0 \times H_0}$  (the index  $H_0 \times H_0$  will henceforth be omitted), where  $u, v \in H_+$ , and  $\bar{v}u$  is the tensor product of the vector  $v$  by  $u$ . The kernel  $K$  is called Hermitian (positive definite) if the form  $B_K$  is Hermitian (positive definite). Our main problem is the representation of a positive definite kernel in the form of a linear combination of elementary positive definite kernels.

Let in  $H_0$  a system of operators  $A^1, \dots, A^p$  be defined with domain  $\mathfrak{D}$ , dense in  $H_0$ . In what follows we shall consider only separable positive spaces and such spaces that contain  $\mathfrak{D}$  as their dense part and contain all  $A^j(\mathfrak{D})$  ( $j = 1, \dots, p$ ). Generalized positive definite kernels  $\varphi_\lambda \in H_- \times H_-$ ,  $\|\varphi_\lambda\|_{H_- \times H_-} \leq 1$  ( $\lambda = (\lambda_1, \dots, \lambda_p)$  is a parameter) will be called elementary with respect to the system of operators  $A^1, \dots, A^p$  if  $(\varphi_\lambda, (A^j - \lambda_{jE})u \cdot \bar{v}) = 0$ ,  $(\varphi_\lambda, \bar{v} \cdot (A^j - \lambda_{jE})u) = 0$  ( $j = 1, \dots, p$ ) for all  $u \in \mathfrak{D}$  and  $v \in H_+$  (from the Hermitian character of the positive definite kernel it follows that one of these equalities entails the other). On the basis of the results of the article <sup>(4)</sup> the following is obtained.

**Theorem 1.** Let  $K \in H_- \times H_-$  be a generalized positive definite kernel; denote by  $H_K$  the completion of  $H_+$  with respect to the scalar product  $\langle u, v \rangle = (K, \bar{v}u)$

$(u, v \in H_+)$ . Suppose first that  $p = 1$ . If the operator  $A^1$  is Hermitian in  $H_K$ :  $\langle A^1 u, v \rangle = \langle u, A^1 v \rangle$  ( $u, v \in \mathfrak{D}$ ), then the representation is always valid

$$K = \int_{-\infty}^{\infty} \varphi_\lambda d\rho(\lambda), \quad (2)$$

where  $\varphi_\lambda \in H_{--} \times H_{--}$  is some family of elementary positive definite kernels with respect to  $A^1$ , and  $d\rho(\lambda)$  is a nonnegative finite measure on the axis; the integral is understood in the weak sense with respect to  $H_{--} \times H_{--}$ . Here  $H_{++} \supset H_+$  is any positive space, chosen in such a way that it is nuclear with respect to  $H_+$ . Conversely, if for the kernel  $K \in H_- \times H_-$  a representation (2) is valid with some family of elementary kernels  $\varphi_\lambda \in H_{--} \times H_{--}$ , then  $A^1$  is Hermitian in  $H_K$ . In the representation (2) the expression  $\varphi_\lambda d\rho(\lambda)$  is determined uniquely if and only if the closure of the operator  $A^1$  is maximal in  $H_K$ .

Let now  $p > 1$ . If the closures of the operators  $A^1, \dots, A^p$  are self-adjoint in  $H_K$  and their resolutions of the identity  $E^1(\Delta_1), \dots, E^p(\Delta_p)$  commute for arbitrary intervals  $\Delta_1, \dots, \Delta_p$ , then the representation (2) is valid with the measure  $d\rho(\lambda)$  in  $p$ -dimensional space and with a family of kernels elementary with respect to  $A^1, \dots, A^p$ . The choice of  $H_{++}$  is the same as above; in the representation (2) the expression  $\varphi_\lambda d\rho(\lambda)$  is determined uniquely. Conversely, if for the kernel  $K \in H_- \times H_-$  a representation (2) is valid with some family of elementary kernels  $\varphi_\lambda \in H_{--} \times H_{--}$ , with  $\varphi_\lambda d\rho(\lambda)$  determined uniquely, then the closures of the operators  $A^1, \dots, A^p$  are self-adjoint in  $H_K$  and their resolutions of the identity commute.

3°. Let us apply this theorem to differential operators  $A^j$ . The case  $p = 1$  has been sufficiently studied (<sup>1,2</sup>); everywhere in what follows we assume  $p > 1$ .

Let  $G$  be a finite or infinite domain of  $n$ -dimensional space;  $H_0 = L_2(G)$ , the involution being ordinary passage to the complex conjugate;  $\mathfrak{D}$  contains the set  $C_0^\infty(G)$  of functions finitely and infinitely differentiable relative to  $G$ , and  $A^j u = \mathcal{L}^j[u]$  ( $u \in C_0^\infty(G)$ ), where  $\mathcal{L}^j$  are linear differential expressions with infinitely differentiable coefficients ( $j = 1, \dots, p$ ). Then the definition of an elementary positive-definite kernel shows that  $\varphi_\lambda$ , in each of the variables, is a generalized solution of a homogeneous linear differential equation, and therefore in the case, for example, of ellipticity of the expressions  $\mathcal{L}^j$  (or when they are in ordinary derivatives) is, inside  $G$ , an ordinary kernel  $\varphi_\lambda(x, y)$  ( $x, y \in G$ ), satisfying the equations  $\mathcal{L}_x^j \varphi_\lambda = \lambda_j \varphi_\lambda$ ,  $\mathcal{L}_y^j \varphi_\lambda = \lambda_j \varphi_\lambda$  ( $j = 1, \dots, p$ ). If  $\mathcal{L}^j$  are expressions in partial derivatives, then these equalities still do not make it possible to express  $\varphi_\lambda$  in terms of a standard system of functions depending only on  $\mathcal{L}^j$  and not depending on  $K$  (cf. (?)). However, in one important case this can be done:

**Theorem 2.** *Let  $G = G_1 \times \dots \times G_n$ , where  $G_j$  is a finite or infinite interval of variation of the variable  $x_j$ ;  $\mathcal{L}^j$  is an ordinary differential expression in the*

variable  $x_j$  of order  $r_j$ ;  $\chi_1^j(x_j, \mu), \dots, \chi_{r_j}^j(x_j, \mu)$  is a fixed fundamental system of solutions of  $\mathcal{L}^j[u] = \mu u$ . Put

$$X_j(x, \lambda) = \chi_{j_1}^1(x_1, \lambda_1) \cdots \chi_{j_n}^n(x_n, \lambda_n) \quad (x = (x_1, \dots, x_n), \lambda = (\lambda_1, \dots, \lambda_n)).$$

Here  $\mathbf{j} = (j_1, \dots, j_n)$  is a combined index, varying over the set  $N$  of points with coordinates  $j_l = 1, \dots, r_l$  ( $l = 1, \dots, n$ ).

Let  $K \in H_- \times H_-$  be a generalized positive-definite kernel; if the closures of the corresponding  $\mathcal{L}^j$ -operators  $A^1, \dots, A^n$  are self-adjoint in  $H_K$  and their resolutions of the identity commute, then the representation

$$K = \int_{-\infty}^{\infty} \sum_{\mathbf{j}, \mathbf{k} \in N} X_j(x, \lambda) \overline{X_k(y, \lambda)} d\rho_{\mathbf{j}\mathbf{k}}(\lambda), \quad (3)$$

holds, where the matrix  $\|d\rho_{\mathbf{j}\mathbf{k}}(\lambda)\|_{\mathbf{j}, \mathbf{k} \in N}$  is positive-definite in the sense that

$$\sum_{\mathbf{j}, \mathbf{k} \in N} \rho_{\mathbf{j}\mathbf{k}}(\Delta) \xi_j \overline{\xi_k} \geq 0 \quad (\rho_{\mathbf{j}\mathbf{k}}(\Delta) = \rho_{\mathbf{j}\mathbf{k}}(\lambda'') - \rho_{\mathbf{j}\mathbf{k}}(\lambda'), \Delta = [\lambda', \lambda'']) \quad (4)$$

for any  $\Delta$  and any set of numbers  $\xi_j$ . The integral in (3) converges in the sense of weak convergence in  $H_- \times H_-$ ;  $H_{++}$  is chosen in the same way as in Theorem 1.

4°. The principal difficulty that arises in applying Theorems 1 and 2 is the verification of the self-adjointness of the operators  $A^j$  and the commutativity of their resolutions of the identity. We shall now formulate (in a general form) the relevant results. Suppose that for each  $j = 1, \dots, n$  there is given a chain  $H_-^j \supset H_0^j \supset H_+^j$  of type (1); assume that the spaces  $H_+$  and  $H_0$  considered earlier have respectively the form  $H_+^1 \times \cdots \times H_+^n$  and  $H_0^1 \times \cdots \times H_0^n$ ; then  $H_- = H_-^1 \times \cdots \times H_-^n$ . Let  $B^j$  be an operator in  $H_+^j$  with domain of definition  $\mathfrak{D}(B^j)$  dense in  $H_+^j$ ; introduce the operator  $A = E \times \cdots \times E \times B^j \times E \times \cdots \times E$  ( $B^j$  stands in the  $j$ -th place) in  $H_+$  with domain of definition

$$\mathfrak{D}(A^j) = H_+^1 \times \cdots \times H_+^{j-1} \times \mathfrak{D}(B^j) \times H_+^{j+1} \times \cdots \times H_+^n,$$

dense in  $H_+$ , and hence also in  $H_0$ . Everywhere below we assume that  $A^j$  is Hermitian in  $H_K$ .

**Theorem 3.** Suppose that for each  $j$ , for both equations  $du_t/dt \pm iB^{j*}u_t = 0$  ( $0 \leq t < \infty$ ), considered in the Hilbert space  $H_+^j$ , uniqueness of weak solutions holds. Then the closures of the operators  $A^j$  are self-adjoint in  $H_K$  and their resolutions of the identity commute.

5°. The scheme indicated in item 4° can be realized for ordinary differential operators  $B^j$  on the whole axis with constant coefficients, acting for different  $j$  in different variables; the proof of uniqueness of solutions in this case is carried out

on the basis of the method developed in [6]. Then, applying Theorem 2, we obtain integral representations of positive-definite kernels. We state the main result, restricting ourselves, for simplicity of formulation, to the case of ordinary positive-definite kernels.

**Theorem 4.** Let in the  $n$ -dimensional space  $E_n$  there be given a continuous positive-definite kernel  $K(x, y)$  ( $x, y \in E_n$ ), satisfying, in the sense of generalized functions, the relations

$$\mathcal{L}_{x_j}^j [K(x, y)] = \overline{\mathcal{L}_{y_j}^j [K(x, y)]} \quad (j = 1, \dots, n), \quad (5)$$

where  $\mathcal{L}^1, \dots, \mathcal{L}^n$  are differential expressions with constant coefficients in the variables, respectively,  $x_1, \dots, x_n$ , and of orders  $r_1, \dots, r_n$ . If for some  $\varepsilon > 0$  the estimate\* holds

$$|K(x_1, \dots, x_n, y_1, \dots, y_n)| \leq C \exp\{|x_1|^{r'_1 - \varepsilon} + |y_1|^{r'_1 - \varepsilon} + \dots + |x_n|^{r'_n - \varepsilon} + |y_n|^{r'_n - \varepsilon}\} \quad (C > 0; x, y \in E_n), \quad (6)$$

where  $r'_j$ , for  $r_j \geq 2$ , is the number conjugate to  $r_j$  ( $1/r_j + 1/r'_j = 1$ ), and for  $r_j = 1$  is an arbitrary positive number, then the representation

$$K(x, y) = \int_{-\infty}^{\infty} \sum_{j, k \in N} X_j(x, \lambda) \overline{X_k(y, \lambda)} d\rho_{jk}(\lambda) \quad (x, y \in E_n) \quad (7)$$

holds, with an absolutely convergent integral. Here the functions  $X_j$  and the matrix  $\|\rho_{jk}\|$  are the same as in Theorem 2.

In this paper we shall not dwell in detail on questions of the uniqueness of the representation (7); we note only that a certain uniqueness of it follows from Theorem 1. It is also clear that the converse assertion to Theorem 4 is valid in the known sense.

From Theorem 4 one can obtain a number of  $n$ -dimensional theorems of Bochner type. Thus, if  $k(x)$  ( $x \in E_n$ ) is a continuous function for which the kernel  $K(x, y) = k(x - y)$  is positive-definite, then (5) is satisfied with  $\mathcal{L}^j = i \partial / \partial x_j$ . Writing the representation (7) for  $K$  and putting  $y = 0$ , we obtain Bochner's theorem (here condition (6) is satisfied automatically). If the kernel  $K(x, y) = k(x + y)$  is positive-definite and  $k(x)$  grows at infinity no faster than  $\exp\{|x_1|^{r'_1} + \dots + |x_n|^{r'_n}\}$ , where  $r'_1, \dots, r'_n$  are some positive numbers, then the representation (7) gives the  $n$ -dimensional theorem of S. N. Bernstein (in this case  $\mathcal{L}^j = \partial / \partial x_j$ ). If the positive-definite kernel  $K(x, y) = k(x + y) + k(x - y)$  and  $k(x)$  grows at infinity no faster than  $\exp\{|x_1|^{2-\varepsilon} + \dots + |x_n|^{2-\varepsilon}\}$  ( $\varepsilon > 0$ ), then one may take  $\mathcal{L}^j = \partial^2 / \partial x_j^2$ ; the representation (7) then gives an expansion of  $k(x)$  in terms of

$\cos \sqrt{\lambda_j} x_j$  and  $\sin \sqrt{\lambda_k} x_k$ . If, in addition, the evenness of  $k(x)$  is known, then in this representation the terms  $\sin \sqrt{\lambda_k} x_k$  vanish, and we obtain a theorem of M. G. Krein type. One could also consider the case when from  $k(x)$  the kernel  $K(x, y)$  is formed by a combination of the indicated operations, for example  $K(x, y) = k(x_1 - y_1, x_2 + y_2) + k(x_1 - y_1, x_2 - y_2)$ . If  $k(x)$  depends analytically on  $x_j$ , then the method given for forming the kernel  $K(x, y)$  can be generalized to the case of expressions  $\mathcal{L}^j$  of order higher than the second.

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- \* The growth of the kernel  $K$  must be somewhat less than the growth of  $\exp\{|x_1|^{r'_1} + |y_1|^{r'_1} + \dots\}$ ; instead of subtracting  $\varepsilon$ , one could have given finer estimates.

*Note: Figure translations are in progress. See original paper for figures.*

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