Corresponding Member of the Academy of Sciences of the USSR G. A. GRINBERG

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Abstract

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Corresponding Member of the Academy of Sciences of the USSR G. A. GRINBERG

ON THE STEADY FLOW OF A CONDUCTING LIQUID THROUGH A RECTANGULAR PIPE LOCATED IN AN EXTERNAL MAGNETIC FIELD, WITH TWO CONDUCTING AND TWO NONCONDUCTING WALLS

1. Shercliff indicated ([1]) that if the external magnetic field $\mathbf{H}^0$ is uniform, and the velocity field and the induced electric and magnetic fields do not depend on the coordinate z, measured in the direction of the axis of the pipe, then there exists a solution of the equations of steady motion of a conducting viscous incompressible liquid through the pipe for which $\mathbf{v} = v\mathbf{i}_z$ and $\mathbf{H} = \mathbf{H}^0 + H_z\mathbf{i}_z$. Choosing the x-axis in the direction of the field $\mathbf{H}^0$, setting $-\partial p/\partial z = P = \text{const}$, and assuming that external body forces are absent, we obtain for H_z and v the following equations:

$$\Delta H_z + \frac{4\pi\mu\sigma H^0}{c^2}\frac{\partial v}{\partial x} = 0,$$

$$\Delta v + \frac{H^0\mu}{4\pi\eta}\frac{\partial H_z}{\partial x} = -\frac{P}{\eta},$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$; η is the coefficient of viscosity of the liquid, and σ and μ are its conductivity and magnetic permeability. The boundary conditions for H_z on the walls s of the pipe* are written as follows: on nonconducting portions of the walls $\partial H_z/\partial s\big|_s = 0$, and on ideally conducting ones $\partial H_z/\partial n\big|_s = 0$, where n is the normal to the wall. To these is added the obvious condition $v\big|_s = 0$.

The case of nonconducting walls was considered in ([1]), and the case of ideally conducting walls in ([2]). In both cases the solution is represented in the form of trigonometric series obtained by the method of particular solutions. In a similar way one could solve the problem for a rectangular pipe whose walls perpendicular to the external magnetic field are ideally conducting and whose walls parallel to it are nonconducting.** It turns out to be much more complicated to consider the question for a pipe in which the walls parallel to $\mathbf{H}^0$ are ideally conducting, and the walls perpendicular to $\mathbf{H}^0$ are nonconducting. Since, as far as we know, an exact solution of this problem has not previously existed in the literature, the present note sets forth some results we have obtained in this direction.

2. Let the coordinates x, y of the vertices of the rectangular cross-section of the pipe be $(0,0)$, $(0,d)$, $(l=2a,0)$, (l,d). Denoting by I the density of the total current entering through the ideally conducting wall $y=0$ and leaving through the wall $y=d$, and introducing into consideration the functions

$$u = \frac{1}{2\gamma}\left[\frac{H^0\mu}{4\pi\eta}\,H_z + \frac{P}{\eta}(x-a)\right],$$

* We regard it as stationary.
** It should be noted that such a form of solution proves very disadvantageous in the case of large values of the Hartmann number, since the convergence of the series sharply worsens as it increases.

where

$$\gamma = \frac{\mu H^0}{2c}\sqrt{\frac{\sigma}{\eta}}$$

and

$$g(\xi,\eta,x,y) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\Big\{K_0\left[\gamma\sqrt{(x_m-\xi)^2+(y_n-\eta)^2}\right] + \\ + K_0\left[\gamma\sqrt{(x'_m-\xi)^2+(y_n-\eta)^2}\right] - K_0\left[\gamma\sqrt{(x_m-\xi)^2+(y'_n-\eta)^2}\right] - \\ - K_0\left[\gamma\sqrt{(x'_m-\xi)^2+(y'_n-\eta)^2}\right]\Big\},$$

where $K_0(z)$ is the Macdonald function, $x_m = 2ml + x$, $x'_m = 2ml - x$, $y_n = 2nd + y$, $y'_n = 2nd - y$. We can write the solution of the problem in the form

$$u(x,y) = \int_0^l [g(\xi,d,x,y) + g(\xi,0,x,y)]f(\xi)\ \operatorname{sh}\gamma(x-\xi)\,d\xi - \alpha\,\frac{\operatorname{sh}\gamma(l-2x)}{\operatorname{sh}\gamma l}, \quad (1)$$

$$v(x,y) = \int_0^l [g(\xi,d,x,y)+$$

$$+g(\xi,0,x,y)]f(\xi)\ \operatorname{ch}\gamma(x-\xi)\,d\xi + 2\alpha\,\frac{\operatorname{sh}\gamma x\ \operatorname{sh}\gamma(l-x)}{\operatorname{sh}\gamma l}, \quad (2)$$

where

$$\alpha = \left(Pl - \frac{H^0 \mu I}{c}\right) \Big/ 4\gamma\eta;$$

$f(\xi)$ is the solution of the integral equation

$$\int_0^l [g(\xi, d, x, 0) + g(\xi, 0, x, 0)] f(\xi) \operatorname{ch} \gamma(x - \xi)\, d\xi = \frac{2\alpha \operatorname{sh} \gamma x \operatorname{sh} \gamma(l - x)}{\operatorname{sh} \gamma l}. \quad (3)$$

For $\gamma d \gg 1$, the term $g(\xi, d, x, 0)$ on the left-hand side of equation (3) becomes very small in comparison with $g(\xi, 0, x, 0)$. Since this term reflects the influence of the finiteness of the size of the pipe cross section in the direction of the y-axis on the values of the sought function $f(\xi)$ along the side $y = 0$, i.e., the influence on this distribution of the wall located at $y = d$, for $\gamma d \gg 1$ this influence may be neglected with the greater accuracy the larger the parameter γd. For $d = \infty$ (the cross section is not in the form of a rectangle, but in the form of a half-strip) it disappears altogether. Equation (3) then takes the form

$$\int_0^l \bar{g}(\xi, 0, x, 0) f_\infty(\xi) \operatorname{ch} \gamma(x - \xi)\, d\xi = \frac{2\alpha \operatorname{sh} \gamma x \operatorname{sh} \gamma(l - x)}{\operatorname{sh} \gamma l}, \quad (4)$$

where $f_\infty(\xi)$ is the corresponding solution, and $\bar{g}(\xi, \eta, x, y)$ is the function obtained from $g(\xi, \eta, x, y)$ for $d = \infty$.

Introducing dimensionless variables $z = x/l$, $\zeta = \xi/l$, and putting $\gamma l = M$ (the Hartmann number), $l f_\infty(zl) = \alpha \psi(z)$, we finally find a one-parameter equation for the function $\psi(z)$:

$$\frac{1}{\pi} \int_0^1 \left\{ \sum_{m=-\infty}^{\infty} [K_0(M|2m + z - \zeta|) - \right.$$

$$\left. - K_0(M|2m - z - \zeta|)] \, \psi(\zeta) \operatorname{ch} M(z - \zeta) \right\} d\zeta = \frac{2 \operatorname{sh} Mz \operatorname{sh} M(1 - z)}{\operatorname{sh} M}. \quad (5)$$

3. For small and moderate values of M, equation (5) can be solved numerically by reducing it to a system of simultaneous linear equations. As regards the case of large M, the series in formula (5) then converges very rapidly, and one may restrict oneself to its first three terms. Formula (5) takes

then the form

$$\frac{1}{\pi}\int_0^1 \{K_0[M|z-\xi|] - K_0[M(z+\xi)] -$$

$$-K_0[M(2-z-\xi)]\}\psi(\xi)\ \mathrm{ch}\, M(z-\xi)\, d\xi = \frac{2\,\mathrm{sh}\, Mz\ \mathrm{sh}\, M(1-z)}{\mathrm{sh}\, M}. \qquad (6)$$

As a more detailed consideration shows, equation (6) can be approximately replaced by

$$\int_0^1 \frac{\chi(\xi)\, d\xi}{\sqrt{|z-\xi|}} = \frac{2\,\mathrm{sh}\, Mz\ \mathrm{sh}\, M(1-z)}{\mathrm{sh}\, M} + e^{-2Mz}\int_z^1 \frac{\chi(\xi)\, d\xi}{\sqrt{z+\xi}} + e^{-2M(1-z)}\int_0^z \frac{\chi(\xi)\, d\xi}{\sqrt{2-z-\xi}}, \qquad (7)$$

where $\chi(\xi) = \psi(\xi)/2\sqrt{2\pi M}$.

The solution of equation (7) may be sought by the method of successive approximations, using the existing methods for solving equations of the form

$$\int_0^1 \frac{\chi(\xi)\, d\xi}{\sqrt{|z-\xi|}} = w(z),$$

where $w(z)$ is a known function (see ([3],[4])).

As the first approximation (to which we shall restrict ourselves here) we take the one in which, in formula (7), the relatively slowly varying factors

$$\int_z^1 \frac{\chi(\xi)\, d\xi}{\sqrt{z+\xi}} \quad \text{and} \quad \int_0^z \frac{\chi(\xi)\, d\xi}{\sqrt{2-z-\xi}}$$

are replaced by their common value*

$$\int_0^1 \frac{\chi(\xi)\, d\xi}{\sqrt{\xi}} = \int_0^1 \frac{\chi(\xi)\, d\xi}{\sqrt{1-\xi}} = A = \mathrm{const}$$

at those points at which the rapidly decreasing exponentials have their maximum value, equal to unity. Equation (7) then takes the following form, if e^{-2M} is neglected in comparison with unity:

$$\int_0^1 \frac{\chi(\xi)\, d\xi}{\sqrt{|z-\xi|}} = 1 + (A-1)\,[e^{-2Mz} + e^{-2M(1-z)}], \qquad (8)$$

where the constant A must be determined from the condition $\chi(0) = 0$. Hence we find that

$$\chi(z) = \chi_1(z) - \frac{\Gamma(3/4)}{\sqrt{\pi}\sqrt[4]{2M}}\,[\chi_2(z) + \chi_2(1-z)],$$

where

$$\chi_1(z) = \frac{1}{\pi\sqrt{2}\,[z(1-z)]^{1/4}}; \tag{9}$$

$$\chi_2(z) = -\frac{1}{4\sqrt{2\pi}\,\Gamma(3/4)(2M)^{1/4}[z(1-z)]^{1/4}} +$$

$$+\frac{\sqrt{2}}{\sqrt{\pi}\,\Gamma^2(3/4)z^{1/4}}\int_0^1 \frac{e^{-2M\sigma}\,[6M\sigma - 8M^2\sigma^2]\,d\sigma}{\sigma^{1/4}}\int_0^{\sqrt[4]{\frac{4}{(1-z)(1-\sigma)}}} \frac{v^2\,dv}{\sqrt{(z-\sigma)^2 + 4z\sigma v^4}}, \tag{10}$$

some terms of order $M^{-5/4}$ having been omitted.

We do not give here a detailed analysis of the formulas obtained. We note only that the function $\chi(z) = \chi(1-z)$, which vanishes at $z = 0$ and at $z = 1$, rises steeply for $M \gg 1$ in the vicinity of the walls, reaching its maximum value $\chi_{\max} \cong 0.268M^{1/4}$ at a distance $\Delta z \cong 0.85/M$ from them, while outside the wall region, whose thickness

* For $\chi(1-\xi) = \chi(\xi)$.

has order $O(1/M)$, and the function $\chi(z)$ is approximately equal to $\chi_1(z)$, since there $|\chi_2(z)| \ll \chi_1(z)$.

Let us also note that the values of $\chi(z)$ found from formulas (9), (10) for $M = 10$ are close to the corresponding quantities obtained by direct numerical solution of equation (6), where $\psi(z) = 2\sqrt{2\pi M}\,\chi(z)$.

Physical-Technical Institute named after A. F. Ioffe
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.