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Abstract

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ON THE REGULARITY OF GENERALIZED SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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In the papers ([1]) we proved the classical solvability "in the large" of the Dirichlet problem for elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x, u, u_{x_k})) + a(x, u, u_{x_k}) = 0, \qquad n \geqslant 2, \tag{1}$$

and investigated the question of the smoothness of generalized solutions of linear and quasilinear elliptic equations of the form (1); in particular, we proved that any bounded generalized solution of the variational problem of finding functions for which the functional

$$I(u) = \int_\Omega F(x, u, u_{x_k})\, dx, \qquad u\,\big|_S = \varphi(s),$$

takes a stationary value belongs to the class $C_{k,\alpha}(\Omega)$, if $F \in C_{k,\alpha}$, $k \geqslant 3$, and if certain "natural requirements" with respect to F are satisfied (see ([1,a,6])). Here we strengthen these results, reducing the smoothness conditions on F and a_i, in a certain respect, to minimal ones. In addition, the method of proving the smoothness of generalized solutions is simpler than that given by us in ([1]).

The main theorems of the present work are the following:

Theorem 1. Let $a_i(x, u, p_k)$ and $a(x, u, p_k)$, as functions of their arguments, belong to the classes $C_{1,\alpha}$, $\alpha > 0$, and $C_{1,0}$, respectively, for $x \in \Omega$, $|u| \leqslant M^*$ and all finite $p = \left(\sum_{k=1}^n p_k^2\right)^{1/2}$. Suppose, moreover, that for the same values of x, u, p_k the conditions

$$|a_i|\, p + |a| \leqslant \mu(1+p)^m, \qquad a_i p_i \geqslant \nu_1 p^m - \mu_1, \qquad m > 1,\ \nu_1 > 0, \tag{2}$$

$$\nu_2(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2 \leqslant \frac{\partial a_i}{\partial p_j}\xi_i\xi_j \leqslant \mu(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2, \qquad \nu_2>0,$$

$$\left|\frac{\partial a_i}{\partial p_j}\right|p^2+\left|\frac{\partial a_i}{\partial u}\right|p+\left|\frac{\partial a}{\partial p_i}\right|p+\left|\frac{\partial a}{\partial u}\right|+\left|\frac{\partial a_i}{\partial x_j}\right|p+\left|\frac{\partial a}{\partial x_i}\right| \leqslant \mu(1+p)^m \tag{3}$$

are fulfilled. Then every generalized solution $u(x)$ of equation (1), i.e. a function from $W_m^1(\Omega)$, satisfying, for an arbitrary bounded $\eta(x)$ from $\overset{\circ}{W}{}_m^1(\Omega)$

* Here and in ([1,3]), u may be regarded as varying in any interval $[M_1, M_2]$, and not only in an interval of the form $[-M, M]$.

the identity

$$\int_\Omega \left[a_i(x,u,u_{x_k})\eta_{x_i} - a(x,u,u_{x_k})\eta\right]\,dx = 0, \tag{4}$$

with $\operatorname{vrai}\max_\Omega |u| \le M$, belongs to the class $C_{2,\alpha}(\Omega')$, where Ω' is any interior subdomain of the domain Ω. If, in addition, $a_i \in C_{l,\alpha}$, and $a \in C_{l-1,\alpha}$, $l>1$, then $u \in C_{l+1,\alpha}(\Omega')$.

If S and $u|_S$ belong to $C_{l+1,\alpha}$, $l \ge 1$, then $u(x)$ belongs to $C_{l+1,\alpha}(\overline{\Omega})$.

A particular, but important, case of Theorem 1 is:

Theorem 2. Let $F(x,u,p_k)$ be defined for $x\in\Omega$, $|u|\le M$, and $|p_k|<\infty$, and satisfy the following "natural" conditions:

1) On any compact set it belongs to the class $C_{l,\alpha}$, $l\ge 2$, $\alpha>0$.

2) F has, with respect to

$$p=\left(\sum_{k=1}^{n}p_k^2\right)^{1/2},$$

growth of order $m>1$, more precisely,

$$\nu_1 p^m \le F(x,u,p_k) \le \mu_1'(1+p)^m, \qquad \nu_1>0,$$

and differentiating F and its derivatives with respect to p_k lowers the orders of their growth in p by at least one, while differentiating with respect to u and x_k does not increase them.

3) The Euler equation for it is uniformly elliptic:

$$\nu_2(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2 \le F_{p_ip_j}(x,u,p_k)\xi_i\xi_j \le \mu_2(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2, \qquad \nu_2>0.$$

4) For sufficiently large p,

$$F_{p_i}(x,u,p_k)p_i > \nu_3 p^m, \qquad p\gg 1, \quad \nu_3>0.$$

Then any function $u(x)$ from $W_m^1(\Omega)$ with $\operatorname{vrai}\max_\Omega |u| \le M$, satisfying the identity

$$\delta I(u) = \int_\Omega \left[F_{u_{x_i}}(x,u,u_{x_k})\eta_{x_i} + F_u(x,u,u_{x_k})\eta\right] dx = 0$$

for arbitrary bounded $\eta(x)$ from $\overset{0}{W}{}_m^1(\Omega)$, belongs to $C_{l,\alpha}(\Omega')$, $\Omega' \Subset \Omega$. If $u|_S$ and the boundary S belong to $C_{l,\alpha}$, then $u(x)\in C_{l,\alpha}(\overline{\Omega})$.

Let us outline the general course of the proof of Theorems 1 and 2.

First we prove (see ($1^{\mathrm{a,b}}$)) that $u(x)$ belongs to the class $C_{0,\beta}$ with some $\beta>0$. Then we establish (see (1^{c})) that $u(x)$ has generalized second-order derivatives and satisfies equation (1) almost everywhere in Ω, and that for $u(x)$ the estimate

$$\int_{\Omega'}\left[|\nabla u|^{m+2} + (1+|\nabla u|)^{m-2}\sum_{i,j=1}^{n} u_{x_ix_j}^2\right] dx \le \text{const} \tag{5}$$

holds (if the boundary data are smooth, then (5) is true for $\Omega'=\Omega$). It is now not difficult to verify, putting in (4) $\eta=\xi_{x_k}$, that $u(x)$ satisfies the integral identity

$$\int_\Omega \left(\frac{da_i}{dx_k}\xi_{x_i} + a\xi_{x_k}\right) dx = 0, \qquad k=1,\dots,n, \tag{6}$$

with any smooth finite function $\xi(x)$. Let $b(x)=\min\{|\nabla u(x)|^2, N^2\}$, and let $\zeta(x)$ be a smooth nonnegative function equal to zero outside the ball $K(\rho)\Subset\Omega$ of radius ρ. In view of conditions (2), (3) and estimate (5), in identity (6)

one may set $\xi(x)=b^r u_{x_k}\zeta^2$, $r\geqslant 1$. Moreover, integration by parts easily verifies the inequality

$$\int_{K(\rho)}(1+|\nabla u|)^m|\nabla u|^2 b^r\zeta^2\,dx \leqslant C\operatorname{osc}\{u,K(\rho)\}\int_{K(\rho)}\left[(1+|\nabla u|)^{m-2}\zeta^2\sum_{i,j=1}^{n}u_{x_ix_j}^2\right.$$

$$+ (1+|\nabla u|)^{m+2}\zeta^2 + (1+|\nabla u|)^m|\nabla\zeta|^2\big]\, b^r\, dx. \tag{7}$$

Using (6) with the indicated ξ, (7), and the fact that u belongs to the class $C_{0,\beta}(\Omega)$, we establish the boundedness of the integrals

$$\int_{\Omega'}\left[|\nabla u|^{m+2r+2} + |\nabla u|^{m+2r-2}\sum_{i,j=1}^{n} u^2_{x_i x_j}\right] dx \leqslant C(r) \tag{8}$$

successively for any $r = 1, 2, \ldots$.

Let now $\zeta(x)$ be a smooth nonnegative finite function in Ω, and let $w(x) = (|\nabla u|^2\zeta^2)^{(m+2)/4}$. We shall denote by A_λ the set of points of the domain Ω for which $w(x) > \lambda$. Then, putting in (6)

$$\xi(x) = \begin{cases} (|\nabla u|^2\zeta^2 - \lambda^{4/(m+2)})/\zeta^m\, u_{x_k}, & |\nabla u|^2\zeta^2 \geqslant \lambda^{4/(m+2)}, \\ 0, & |\nabla u|^2\zeta^2 \leqslant \lambda^{4/(m+2)}, \end{cases} \qquad k = 1, \ldots, n,$$

and using condition (3), we obtain, for any $\lambda \geqslant 0$, the inequality

$$\int_{A_\lambda} |\nabla w|^2\, dx \leqslant C\int_{A_\lambda} (1+|\nabla u|)^{m+4}(1+|\nabla\zeta|^2)\, dx. \tag{9}$$

Taking into account the boundedness of the integrals (8), one can estimate the right-hand side of (9) by Hölder' s inequality, so that

$$\int_{A_\lambda} |\nabla w|^2\, dx \leqslant C_1 \operatorname{mes}^{1-\varepsilon} A_\lambda, \qquad \varepsilon < \frac{2}{n}.$$

Hence (see $(1^{\mathrm{b}}, 2)$) the boundedness of $w(x)$ follows, and therefore also that of $|\nabla u|$ in any $\Omega' \Subset \Omega$. Further, applying Morrey' s theorem, which we proved for any $n \geqslant 2$ $(3, 1^{\mathrm{b}})$, we are convinced of the validity of the first part of Theorem 1.

To prove the assertion of the theorem for a closed domain $\overline{\Omega}$, we first estimate $M_1 = \operatorname{vrai\,max}_S |\nabla u|$, using the method of auxiliary functions of S. N. Bernstein and the fact that inside Ω, $u'_{x_i}(x)$ is a smooth function. After this, in obtaining the estimates (8) and (9), instead of $|\nabla u|$ one must take

$$z(x) = \begin{cases} |\nabla u| - M_1, & |\nabla u| \geqslant M_1, \\ 0, & |\nabla u| \leqslant M_1, \end{cases}$$

and regard $\zeta(x)$ as not equal to zero on some part of the boundary S.

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Note: Figure translations are in progress. See original paper for figures.