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Abstract

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SPACES OF FUNCTIONS ANALYTIC IN (p, q)-CIRCULAR DOMAINS

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Let Q be a bounded simply connected domain of the space C^n of n complex variables. Denote by $A(Q)$ and $A(\overline{Q})$ the spaces of functions analytic, respectively, in the domain Q and in $\overline{Q}$. The topology in the spaces $A(Q)$ and $A(\overline{Q})$ is introduced in the generally known way (see, for example, ([1])). $A(Q)$ is a space of type F, $A(\overline{Q})$ a space (LN^*) ([1]). It is known that simply connected domains Q and Q_1 of the space C^n, $n > 1$, generally speaking, are not pseudoconformally mapped onto one another ([2], Ch. VIII). Therefore the question of the isomorphism of the spaces $A(Q)$ and $A(Q_1)$ (respectively $A(\overline{Q})$ and $A(\overline{Q_1})$) is nontrivial.

An isomorphism of spaces of functions analytic in a hypercone and a bicylinder was in fact obtained earlier by A. A. Temlyakov ([3]). In ([4]) an isomorphism was proved for spaces of functions analytic in bounded polycircular domains containing their center. Similar results, as became known to us, were obtained by C. Rolewicz ([17]) and C. D. Okunev. In the present note an isomorphism is proved for spaces of functions analytic in bounded circular ([5], p. 109) domains containing their center. For simplicity the exposition is given for the case of two complex variables.

1. Let D be a bounded (p, q)-circular domain ([2], p. 117) with center at the origin of the space C^2 of complex variables (w, z), where p, q are relatively prime positive integers. We shall assume that the domain D contains its center. Without loss of generality one may suppose that the domain D is complete ([2 5]). We shall also require that the domain

$$\widetilde{D} = \{(e^{i\arg w}|w|^p,\ e^{i\arg z}|z|^q):\ (w, z) \in D\}$$

be such that the length of the segment of a ray connecting the center of the domain $\widetilde{D}$ with a boundary point is a continuous function of the position of the ray.

Denote by $d\omega$ the volume element of four-dimensional space. The integral over the domain D will, when necessary, be understood as improper ([[2]], p. 119).

Lemma 1. Let a function $f(w, z)$ analytic in the domain D be representable by a series of polynomials uniformly convergent inside D,

$$f(w, z) = \sum_{k=0}^{\infty} P_k(w, z),$$

where $P_k(w, z)$ are homogeneous polynomials of degree k with respect to $w^{1/p}, z^{1/q}$. For the existence of the integral

$$I(f) = \int_D |f(w, z)|^2 \, d\omega$$

it is necessary and sufficient that the series

$$\sum_{k=0}^{\infty} \int_D |P_k(w, z)|^2 \, d\omega.$$

converge. The sum of this series is equal to $l(f)$.

This lemma is a generalization of A. Cartan' s well-known theorem for circular domains ([6]). The proof is analogous to the proof of A. Cartan' s theorem.

Introduce the following sets: $L(k, p, q) = \{m$: there exists an n such that $mp + nq = k\}$, $M(p, q) = \{(k, m) : m \in L(k, p, q),\ L(k, p, q)$ is nonempty$\}$, where m, n, k are nonnegative integers.

Lemma 2. In the space $A(D)$ there exists a basis consisting of polynomials in w, z:

$$\{P_{km}(w, z)\}_{(k,m) \in M(p,q)},$$

where $P_{km}(w, z)$ are homogeneous polynomials of degree k with respect to $w^{1/p}, z^{1/q}$.

Proof. Taking into account that for $mp + nq \neq m_1 p + n_1 q$

$$\int_D w^m z^n \bar{w}^{m_1} \bar{z}^{n_1} \, d\omega = 0,$$

and orthonormalizing, for a given k, the monomials

$$\{w^m z^n\}_{mp+nq=k},$$

we obtain a system of polynomials orthonormal in the domain D,

$$\{P_{km}(w,z)\}_{(k,m)\in M(p,q)}, \tag{1}$$

where $P_{km}(w,z)$ are homogeneous polynomials of degree k with respect to $w^{1/p}, z^{1/q}$. From Lemma 1 and one theorem of A. Cartan ([6]) it follows that the system (1) is closed in the space $L^2(D)$ of functions analytic in the domain D and square-summable in the domain D. Therefore every function from $L^2(D)$ expands into a series converging uniformly inside the domain D ([2], p. 123):

$$\sum_{(k,m)\in M(p,q)} a_{km}P_{km}(w,z). \tag{2}$$

Put $D_r = \{(w,z) : (w/r^p,\ z/r^q) \in D\}$, $r > 0$. It is easy to see that if the system (1) is orthonormal in the domain D, then the system of polynomials

$$\left\{\frac{1}{r^{k+p+q}}P_{km}(w,z)\right\}_{(k,m)\in M(p,q)}$$

is orthonormal in the domain D_r.

Let $f(w,z) \in A(D)$. Since $f(w,z) \in L^2(D_r)$ for all $r < 1$, there exist coefficients $a_{km}^{(r)}$ such that the function $f(w,z)$ is representable, by a series uniformly convergent in the domain D_r,

$$f(w,z) = \sum_{(k,m)\in M(p,q)} \frac{a_{km}^{(r)}}{r^{k+p+q}}P_{km}(w,z).$$

From the uniqueness of the expansion of $f(w,z)$ into a multiple power series in a neighborhood of the point $(0,0)$, it follows that for any $\rho, r < 1$

$$\frac{a_{km}^{(r)}}{r^{k+p+q}} = \frac{a_{km}^{(\rho)}}{\rho^{k+p+q}}.$$

Consequently, the function $f(w,z)$ expands in a series of the form (2), uniformly convergent inside the domain D. Lemma 2 is proved.

In what follows we shall assume that the system of polynomials (1) is a basis in the space $A(D)$. Put

$$d(D;P_{km}) = \sup_{(w,z)\in D} |P_{km}(w,z)|.$$

Theorem 1. *In order that the series*

$$\sum_{(k,m)\in M(p,q)} a_{km} P_{km}(w,z) \tag{3}$$

converge uniformly inside the domain D, it is necessary and sufficient that, in the unit bicylinder

$$E_{1,1} = \{(w,z) : |w| < 1,\ |z| < 1\},$$

the series

$$\sum_{m,n=0}^{\infty} a_{km} d(D; P_{km}) w^{mp} z^{nq}, \tag{4}$$

converge, where $k = mp + nq$.

Proof. Necessity. The space $A(D)$ is nuclear (see, for example, (7)). From Lemma 2 and the theorem of A. S. Dynin–B. S. Mityagin ([8,16]) on bases in nuclear spaces it follows that the series

$$\sum_{(k,m)\in M(p,q)} |a_{km}|\, d(D_r; P_{km}) \tag{5}$$

converges for all $r < 1$. Hence, and from the equality

$$d(D_r; P_{km}) = r^k d(D; P_{km}),$$

we obtain that the series (4) converges in the bicylinder $E_{1,1}$.

Sufficiency. From the convergence of the series (4) in the domain $E_{1,1}$ it follows that the series (5) converges for any $r < 1$, i.e. the series (3) converges in the domain D "normally" (([2]), p. 278), and hence also uniformly.

Corollary 1. *In order that some sequence $\{\alpha_{km}\}$ have the property of the sequence $\{d(D; P_{km})\}$ indicated in Theorem 1, it is necessary and sufficient that the equality*

$$\lim_{k\to\infty} \sqrt[k]{\frac{d(D; P_{km})}{|\alpha_{km}|}} = 1$$

hold.

Consider the following problem: to determine the greatest r, $0 \le r \le \infty$, such that the series (3) converges uniformly inside the domain D_r, where D is a fixed domain.

Corollary 2. *The greatest r, $0 \le r \le \infty$, such that the series (3) converges uniformly inside the domain D_r, is determined by the formula (we assume that $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$)*

$$\frac{1}{r} = \lim_{k\to\infty} \sqrt[k]{|a_{km}|\, d(D; P_{km})}.$$

Remark 1. Theorem 1 for the case of the hypercone

$$R = \{(w, z) : |w| + |z| < 1\}$$

was proved by A. A. Temlyakov ([3]). In the case of complete circular domains, Theorem 1 was obtained in ([4]). Corollary 2 for the case of a double power series and a domain D that is a hypercone was also indicated by A. A. Temlyakov ([3]).

2. We introduce the countably normed ([9]) space $B(D)$ of sequences

$$a = \{a_{km}\}_{(k,m)\in M(p,q)}$$

with the system of norms

$$\|a\|_r = \sum_{(k,m)\in M(p,q)} |a_{km}|\, d(D; P_{km})\, r^k, \qquad r < 1.$$

With the aid of Theorem 1 and Banach' s theorem ([10], p. 56), it is easy to obtain the following lemma:

Lemma 3. The spaces $A(D)$ and $B(D)$ are isomorphic.

Let D and D_1 be domains satisfying the conditions of item 1.

Theorem 2. The spaces $A(D)$ and $A(D_1)$ are isomorphic. The spaces $A(\overline{D})$ and $A(\overline{D}_1)$ are also isomorphic.

Remark 2. With the help of the results obtained, one can establish the general form of linear continuous functionals in the space $A(D)$ (in the space $A(\overline{D})$). For bicircular domains this was done in the papers ([4], [11]).

Remark 3. Theorem 2 makes it possible to reduce questions of completeness and bases in the spaces $A(D)$ to analogous questions in the space $A(E_{1,1})$. Thus the results of the papers ([12–15]) automatically extend to the spaces $A(D)$.

Remark 4. The entire content of the present note carries over to the case of (p, q)-circular domains, where p, q are relatively prime integers, $pq > 0$.

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Note: Figure translations are in progress. See original paper for figures.