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Abstract

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MATHEMATICS

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ON THE EMBEDDING IN E^4 OF CERTAIN MONOTONE IMAGES OF E^3

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1. Bing constructed a continuous decomposition E_f^3 of three-dimensional Euclidean space E^3, whose elements are points and tame simple arcs* and whose space H is not embeddable in E^3 ([2]). The set of nondegenerate (to a point) elements of the decomposition E_f^3 is zero-dimensional and compact. Curtis indicated a sufficient condition for the embeddability of a monotone image of E^k in E^n and showed ([5]), on its basis, that H is topologically embeddable in E^4, while Bing showed ([8]) that $E^4 = H \times E^1$, where E^1 is a line. Here we indicate a sufficient criterion for the topological embeddability of a monotone image of E^3 in E^4, different from Curtis' s criterion, from which it follows directly that H is embeddable in E^4.

We consider a continuous decomposition of E^3 for which the closure of the set of all nondegenerate elements of the decomposition is zero-dimensional and compact, and each nondegenerate element ξ is a smooth continuum; we call a continuum $K \subset E^3$ smooth if $E^3 \setminus K$ is homeomorphic to the complement of E^3 to a point. Bing showed that an arbitrary topological sphere S^2 in E^3 can be approximated as closely as desired by a polyhedral sphere ([3]). By Aleksander' s theorem ([1]), for a polyhedral sphere there exists a homeomorphic mapping of E^3 onto itself under which S^2 is carried into the sphere $x^2 + y^2 + z^2 = 1$, and by Moise' s theorem ([7]), for any two finite sequences of polyhedral spheres in E^3

$$S_1 \supset S_2 \supset \dots \supset S_n; \qquad S_1' \supset S_2' \supset \dots \supset S_n'$$

there exists a homeomorphic mapping $\varphi : E^3 \to E^3$, under which $\varphi(S_k) = S_k'$, $k = 1, 2, \dots, n$. On the basis of these three theorems it is easy to show that the condition of smoothness of a continuum K in E^3 is equivalent to the condition: for any $\varepsilon > 0$ there exists a neighborhood of the continuum K of diameter $< \varepsilon$, whose closure is a polyhedral ball.

2. Denote by $P = \{\xi\}$ the set of all nondegenerate elements of some continuous decomposition of E^3, and by $P^* = \bigcup \xi$ the set-theoretic sum of these elements; by A_f denote the continuous decomposition of the space A, and by $f(A)$ the space of this decomposition.

Theorem 1. *If E^3_f is a continuous decomposition of E^3 into points and smooth continua such that the closure $\bar{P}$ of the set P of all nondegenerate elements of the decomposition is zero-dimensional and compact, then the space of this decomposition $f(E^3)$ is topologically embeddable in E^4.*

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* A simple arc $l \subset E^3$ is called **tame** if there exists a homeomorphic mapping $\varphi : E^3 \to E^3$ under which l is carried into a segment of a straight line.

The proof of Theorem 1 is based on two lemmas.

Lemma 1. Let $E^4 = E^3 \times E^1$ and $W = \Delta \times [a, b]$, where Δ is a tame topological ball in E^3, and $[a, b]$ is a segment of E^1. For any number $\varepsilon > 0$ and segment $[a', b']$, $a < a' < b' < b$, there exist two tame topological balls in E^3

$$\Delta^* \subset \tilde{\Delta} \subset \bar{\Delta} \subset \Delta$$

such that*

$$\operatorname{diam} \Delta^* < \varepsilon; \qquad h(\Delta, \tilde{\Delta}) < \varepsilon,$$

and a topological mapping

$$\varphi : \bar{W} \to \bar{W},$$

fixed on the boundary of W and carrying $\tilde{W} = \tilde{\Delta} \times [a', b']$ into $W^* = \Delta^* \times [a', b']$, such that

$$\varphi(M, t) = [M', t],$$

where $M \in \Delta$, $M' \in \Delta$, $t \in E^1$, $a \leq t \leq b$.

Lemma 2. If E^3_f is a continuous decomposition of E^3 satisfying the conditions of Theorem 1, then the closure of the sum P^* of all nondegenerate elements of the decomposition can be represented in the form

$$\text{a)} \quad \bar{P}^* = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{N_k} U^k_n; \qquad \text{b)} \quad \bar{P}^* = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{M_k} \Delta^k_m, \tag{1}$$

where each U_n^k is a domain consisting of whole elements of the decomposition E_f^3; the boundary of U_n^k does not intersect $\bar{P}^*$, and

$$\bar{U}_n^k \cap \bar{U}_{n'}^k = \Lambda \quad \text{for } n \neq n'. \tag{2}$$

Each $\bar{\Delta}_m^k$ is a polyhedral ball, and for each U_n^k there are U_r^{k-1} and Δ_m^k such that

$$\bar{U}_m^k \subset \Delta_m^k \subset \bar{\Delta}_m^k \subset U_r^{k-1} \tag{3}$$

and any $\xi \in P$ can be represented in the form

$$\xi = \bigcap_{k=1}^{\infty} U_{n_k}^k; \qquad \bar{U}_{n_k}^k \subset U_{n_{k-1}}^{k-1}. \tag{4}$$

It follows from condition (2) that the representation (4) is unique. By virtue of (3), ξ can also be represented as the intersection of the polyhedral balls Δ_m^k; however, the balls Δ_m^k and $\Delta_{m'}^k$ may intersect, and therefore such a representation of ξ is not unique.

3. The idea of the proof of Theorem 1. We first construct in E^4 a surface L, given by a continuous function

$$t = F(M), \qquad M \in E^3, \quad t \in E^1,$$

such that $F(M)$ is constant on each continuum $\xi \in \bar{P}$. To this end, to each U_n^k we assign a segment $\pi_n^k \subset E^1$ so that

$$\bar{\pi}_i^k \cap \bar{\pi}_j^k = \Lambda, \quad \text{if } \bar{U}_i^k \cap \bar{U}_j^k = \Lambda;$$

$$\bar{\pi}_r^{k+1} \subset \pi_s^k, \quad \text{if } \bar{U}_r^{k+1} \subset U_s^k,$$

and the lengths of π_n^k tend to zero together with $1/k$.

* By $h(A, B)$ we denote the Hausdorff distance between the sets A and B.

Set

$$F(M) = t_\xi, \qquad \text{if } M \in \xi,$$

where

$$\xi = \bigcap_{k=1}^{\infty} U_n^k, \qquad t_\xi = \bigcap_{k=1}^{\infty} \pi_n^k.$$

We extend F to a continuous function on E^3 and denote by L the graph of F.

Next we choose a sequence of numbers $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and construct a sequence of homeomorphic ε_n-shifts of E^4 onto itself: $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$, so that for the homeomorphism $\Psi_n = \varphi_n \varphi_{n-1} \cdots \varphi_1$ and for each $\xi \in \overline{P}$ one has

$$\operatorname{diam} \Psi_n(\xi \times t_\xi) < \varepsilon_n.$$

Each φ_n does not change the coordinate t of a point $(M, t) \in E^4$ and is fixed outside $\bigcup \Psi_{n-1}(W_i^k)$, where $W_i^k = \pi_i^k \times \Delta_m^k$, and Δ_m^k is some polyhedral ball satisfying condition (3) for U_i^k, chosen from the system constructed on the basis of Lemma 2.

The sequence of homeomorphisms Ψ_n converges uniformly to a continuous mapping $\Phi : E^4 \to E^4$; the only nondegenerate elements of the decomposition E_Φ^4 are the continua $\xi \times t_\xi$. Therefore $\Phi(L)$ is homeomorphic to $f(E^3)$ and $\Phi(L) \subset E^4$.

Let us note that it is not known whether, in the formulation of Theorem 1, the condition that $\overline{P}$ be zero-dimensional can be replaced by the weaker condition that P be zero-dimensional.

4. From the sufficient condition for the embeddability of a monotone image of E^3 in E^3, given by Harrold ([6]), it follows directly that if in formula (1) all U_n^k are topological balls, then $f(E^3) \cong E^3$.

In Bing' s example ([2]) of a decomposition space of E^3 not embeddable in E^3, the set of all nondegenerate elements of the decomposition is representable by formula (1), where each U_n^k is a ball with two handles. Moreover, this continuous decomposition is completely continuous on P^*, i.e. the corresponding mapping f is open on P^*.

Bing constructed another continuous decomposition of E^3 into points and a zero-dimensional perfect set P of tame simple arcs, for which the decomposition space coincides with E^3 ([4]). Here each $\overline{U}_n^k$ is a solid torus, i.e. a topological ball with one handle; the decomposition is completely continuous on P. The question arises: can one construct a continuous decomposition of E^3 into points and a zero-dimensional perfect set of tame continua so that all $\overline{U}_n^k$ are solid tori and the decomposition space is not embeddable in E^3?

Such an example can be constructed by specifying the set P so that in each solid torus $\overline{U}_n^{k-1}$, $k \geq 1$, there are contained three solid tori $\overline{U}_1^k, \overline{U}_2^k, \overline{U}_3^k$, arranged so that each of them lies in a topological cube contained in $\overline{U}_n^{k-1}$, but no sum of two distinct $\overline{U}_i^k$, $i = 1, 2, 3$, can be enclosed in a topological cube contained in $\overline{U}_n^{k-1}$.

Then the decomposition space is not embeddable in E^3. The nondegenerate elements of the decomposition ξ are tame continua, among which there is an uncountable set of indecomposable continua; the decomposition is completely continuous on P^*.

At the same time the following holds.

Theorem 2. Let E^3_f be a continuous decomposition of E^3 into points and a zero-dimensional compact set P of tame locally connected continua. If the sum P^* of all nondegenerate elements of the decomposition is representable in the form (1a) in such a way that all $\overline{U}^k_n$ are solid tori and the decomposition E^3_f is completely continuous on P^*, then the decomposition space coincides with E^3.

The proof of Theorem 2 is based on the following lemma:

Lemma 3. *Under the hypotheses of Theorem 2, whatever the number $\varepsilon > 0$ and the integer $r > 0$, there exists a number $R > r$ and a homeomorphic mapping Φ of the space E^3 onto itself, fixed outside $\bigcup_{n=1}^{R} U'_n$, and such that for each $\overline{U^R_i}$*

$$\operatorname{diam} \Phi(\overline{U^R_i}) < \varepsilon.$$

For the proof we first show that among all possible $\overline{U^k_n}$ in $\{1a\}$ there is **at least one** $\overline{U^{k_0}_{n_0}}$, possessing certain special properties, such that for $\overline{U^{k_0}_{n_0}} \cap P^*$ Lemma 3 can be proved by the method used by Bing in proving the analogous lemma in ([4]). After this, Lemma 3 is proved by means of transfinite induction.

It follows from Theorem 2 that if one requires complete continuity of the decomposition on P^*, as is the case in Bing' s examples, then it is **impossible** to construct an upper semicontinuous decomposition of E^3 into points and a null perfect set of tame (and even smooth) simple arcs in such a way that all $\overline{U^k_n}$ are full tori, while the decomposition space is not embeddable in E^3.

It is unknown, however, whether this nevertheless cannot be done if the condition of complete continuity of the decomposition on P^* is dropped.

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Note: Figure translations are in progress. See original paper for figures.