ON THE QUESTION OF COMPUTING EIGENVALUES AND EIGENVECTORS OF MATRICES

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Abstract

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ON THE QUESTION OF COMPUTING EIGENVALUES AND EIGENVECTORS OF MATRICES

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In paper (1) we proposed a method for the approximate computation of eigenvalues and eigenvectors of matrices, based on the use of the parameter-variation method. In the present paper we generalize this method to the case of complex eigenvalues of real matrices. In doing so, differential equations are constructed which are satisfied by the real and imaginary parts of an eigenvalue of a matrix. These equations make it possible to determine, according to one and the same computational scheme, both complex eigenvalues and the corresponding eigenvectors, as well as real ones.

1°. Let a square matrix $A(\lambda) = \|a_{kl}(\lambda)\|$ $(k, l = 1, 2, \dots, n)$ of order n be given, whose elements are functions of a parameter λ taking prescribed values on some finite interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$.

Suppose that the eigenvalue $p_j(\lambda)$ $(j = 1, 2, \dots, n)$ of the matrix $A(\lambda)$ that interests us is complex on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$:

$$p_j(\lambda) = p_{j0}(\lambda) + ip_{j1}(\lambda),$$

and that for some prescribed value of the parameter λ from this interval, for example $\lambda = \lambda_0$, the value of the eigenvalue $p_j(\lambda)$ is known to us:

$$\text{for } \lambda = \lambda_0 \qquad p_{j0}(\lambda) = p_{j0}^{(0)}, \quad p_{j1}(\lambda) = p_{j1}^{(0)}. \tag{1}$$

Suppose, moreover, that:

1) all the functions $a_{kl}(\lambda)$ are defined and continuous on the entire interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$ and have continuous derivatives on this interval;
2) the trace of the adjoint matrix $C(\lambda, p_{j0}, p_{j1})$ to the matrix $\|A(\lambda) - (p_{j0} + ip_{j1})E\|$ is nonzero at the point $(\lambda_0, p_{j0}^{(0)}, p_{j1}^{(0)})$.

It is required to find approximate values of the eigenvalue $p_j(\lambda) = p_{j0}(\lambda) + ip_{j1}(\lambda)$ for prescribed values of the parameter $\lambda > \lambda_0$.

For this purpose we proceed as follows* . Differentiating with respect to λ the equation

$$\omega(\lambda, p_0, p_1) = \operatorname{Det} \|A(\lambda) - (p_0 + ip_1)E\| = 0, \tag{2}$$

we obtain

$$\frac{\partial \omega(\lambda, p_0, p_1)}{\partial p_0} \frac{dp_0}{d\lambda} + \frac{\partial \omega(\lambda, p_0, p_1)}{\partial p_1} \frac{dp_1}{d\lambda} + \frac{\partial \omega(\lambda, p_0, p_1)}{\partial \lambda} = 0.$$

The last equation, by virtue of the lemma proved in (2), can be rewritten in the form

$$\operatorname{Sp} C(\lambda, p_0, p_1) \frac{dp_0}{d\lambda} + i \operatorname{Sp} C(\lambda, p_0, p_1) \frac{dp_1}{d\lambda} = \operatorname{Sp} \left[C(\lambda, p_0, p_1) \frac{dA(\lambda)}{d\lambda} \right]. \tag{3}$$

* In what follows, for simplicity of notation, the subscript j is omitted.

Suppose that the trace of the matrix $C(\lambda, p_0, p_1)$ is different from zero at all points of the domain G of variation of λ, p_0, and p_1, containing the point $(\lambda_0, p_0^{(0)}, p_1^{(0)})$, i.e.,

$$\operatorname{Sp} C(\lambda, p_0, p_1) \neq 0 \quad \text{in } G. \tag{4}$$

Let us write the matrix $\|A(\lambda) - (p_0 + ip_1)E\|$ in the form

$$\|A(\lambda) - (p_0 + ip_1)E\| = \begin{Vmatrix} L(\lambda, p_0, p_1) & u(\lambda) \\ v(\lambda) & a_{n,n}(\lambda) - (p_0 + ip_1) \end{Vmatrix},$$

where

$$L(\lambda, p_0, p_1) = L_0(\lambda, p_0) - ip_1 E,$$

$$L_0(\lambda, p_0) = \begin{Vmatrix} a_{11}(\lambda) - p_0 & a_{12}(\lambda) & \cdots & a_{1,n-1}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) - p_0 & \cdots & a_{2,n-1}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1}(\lambda) & a_{n-1,2}(\lambda) & \cdots & a_{n-1,n-1}(\lambda) - p_0 \end{Vmatrix},$$

$$v(\lambda) = \{a_{n,1}(\lambda), a_{n,2}(\lambda), \dots, a_{n,n-1}(\lambda)\}, \qquad u(\lambda) = \begin{pmatrix} a_{1,n}(\lambda) \\ a_{2,n}(\lambda) \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1,n}(\lambda) \end{pmatrix}.$$

We shall assume that the determinant $\bar{\Delta}(\lambda, p_0, p_1)$ of the matrix $L(\lambda, p_0, p_1)$ is different from zero in the domain G. This can always be achieved by permuting the corresponding rows and columns of the matrix $\|A(\lambda) - (p_0 + ip_1)E\|$.

Proceeding now analogously to how we did this in (2), the adjugate matrix $C(\lambda, p_0, p_1)$ can be written on the interval $\lambda_0 \le \lambda \le \lambda^*$ in the following form:

$$C(\lambda, p_0, p_1) = \bar{\Delta}(\lambda, p_0, p_1)\, C^*(\lambda, p_0, p_1).$$

Here

$$C^*(\lambda, p_0, p_1) = \left\| \begin{array}{cc} F(\lambda, p_0, p_1) & -L^{-1}(\lambda, p_0, p_1)u(\lambda) \\ -v(\lambda)L^{-1}(\lambda, p_0, p_1) & 1 \end{array} \right\|,$$

$$F(\lambda, p_0, p_1) = L^{-1}(\lambda, p_0, p_1)u(\lambda)v(\lambda)L^{-1}(\lambda, p_0, p_1).$$

The matrix $L^{-1}(\lambda, p_0, p_1)$ can be represented in the form

$$L^{-1}(\lambda, p_0, p_1) = [L_0(\lambda, p_0) + ip_1 E]\, M^{-1}(\lambda, p_0, p_1),$$

where the matrix $M(\lambda, p_0, p_1)$ is determined by the formula

$$M(\lambda, p_0, p_1) = L_0^2(\lambda, p_0) + p_1^2 E.$$

Then the matrix $F(\lambda, p_0, p_1)$ is rewritten in the following form:

$$F(\lambda, p_0, p_1) = F_0(\lambda, p_0, p_1) + ip_1 F_1(\lambda, p_0, p_1),$$

where

$$\begin{gathered} F_0(\lambda, p_0, p_1) = Qu(\lambda)v(\lambda)Q - p_1^2 M^{-1}u(\lambda)v(\lambda)M^{-1}, \\ F_1(\lambda, p_0, p_1) = M^{-1}u(\lambda)v(\lambda)Q + Qu(\lambda)v(\lambda)M^{-1}, \\ Q = Q(\lambda, p_0, p_1) = L_0(\lambda, p_0)M^{-1}(\lambda, p_0, p_1), \\ M = M(\lambda, p_0, p_1). \end{gathered}$$

Thus, the matrix $C^*(\lambda, p_0, p_1)$ takes the form

$$C^*(\lambda, p_0, p_1) = C_0^*(\lambda, p_0, p_1) + ip_1 C_1^*(\lambda, p_0, p_1),$$

where

$$C_0^*(\lambda, p_0, p_1) = \left\| \begin{array}{cc} F_0(\lambda, p_0, p_1) & -Q(\lambda, p_0, p_1)u(\lambda) \\ -v(\lambda)Q(\lambda, p_0, p_1) & 1 \end{array} \right\|,$$

$$C_1^*(\lambda, p_0, p_1) = \left\| \begin{array}{cc} F_1(\lambda, p_0, p_1) & -M^{-1}(\lambda, p_0, p_1)u(\lambda) \\ -v(\lambda)M^{-1}(\lambda, p_0, p_1) & 0 \end{array} \right\|.$$

In this case equation (3) is equivalent to the following two equations:

$$\mathrm{Sp}\, C_0^* \frac{dp_0}{d\lambda} - p_1 \,\mathrm{Sp}\, C_1^* \frac{dp_1}{d\lambda} = \mathrm{Sp}\left(C_0^* \frac{dA(\lambda)}{d\lambda} \right),$$

$$p_1 \,\mathrm{Sp}\, C_1^* \frac{dp_0}{d\lambda} + \mathrm{Sp}\, C_0^* \frac{dp_1}{d\lambda} = p_1 \,\mathrm{Sp}\left(C_1^* \frac{dA(\lambda)}{d\lambda} \right), \tag{5}$$

where, to shorten the notation, we have set

$$C_0^* = C_0^*(\lambda, p_0, p_1), \qquad C_1^* = C_1^*(\lambda, p_0, p_1).$$

Solving equations (5) with respect to $dp_0/d\lambda$ and $dp_1/d\lambda$, we find

$$\frac{dp_0}{d\lambda} = \frac{\mathrm{Sp}\, C_0^* \,\mathrm{Sp}(C_0^* dA(\lambda)/d\lambda) + p_1^2 \,\mathrm{Sp}\, C_1^* \,\mathrm{Sp}(C_1^* dA(\lambda)/d\lambda)}{(\mathrm{Sp}\, C_0^*)^2 + p_1^2 (\mathrm{Sp}\, C_1^*)^2},$$

$$\frac{dp_1}{d\lambda} = p_1 \frac{\mathrm{Sp}\, C_0^* \,\mathrm{Sp}(C_1^* dA(\lambda)/d\lambda) - \mathrm{Sp}\, C_0^* \,\mathrm{Sp}(C_0^* dA(\lambda)/d\lambda)}{(\mathrm{Sp}\, C_0^*)^2 + p_1^2 (\mathrm{Sp}\, C_1^*)^2}. \tag{6}$$

To determine now, for given λ, approximate values of the eigenvalue $p(\lambda)$ of the matrix $A(\lambda)$, we numerically integrate equations (6) on the interval $\lambda_0 \le \lambda \le \lambda^*$ with initial conditions (1). In this case each column of the matrix $C^*(\lambda, p_0, p_1)$ will consist of components of the eigenvector $X(\lambda)$ belonging to the eigenvalue $p(\lambda)$.

The values of the elements of the inverse matrix $M^{-1}(\lambda, p_0, p_1)$ at each step can also be computed by the parameter-variation method [3], provided that the matrix $M^{-1}(\lambda, p_0, p_1)$ is known at the point $(\lambda_0, p_0^{(0)}, p_1^{(0)})$:

$$\text{for } \lambda = \lambda_0 \qquad p_0(\lambda) = p_0^{(0)}, \quad p_1(\lambda) = p_1^{(0)}, \quad M^{-1}(\lambda, p_0, p_1) = M_0^{-1}. \tag{7}$$

In this case the entire computation process is reduced to the numerical integration on the interval $\lambda_0 \le \lambda \le \lambda^*$ of equations (6) together with the matrix equation

$$\frac{dM^{-1}(\lambda, p_0, p_1)}{d\lambda} = -M^{-1}(\lambda, p_0, p_1) \frac{dM(\lambda, p_0, p_1)}{d\lambda} M^{-1}(\lambda, p_0, p_1) \tag{8}$$

under the initial conditions (7).

Above we assumed that condition (4) is fulfilled. Note that if the eigenvalue $p(\lambda)$ is simple on the interval $\lambda_0 \le \lambda \le \lambda^*$, then condition (4) will be fulfilled for any λ from the interval under consideration.

Of particular interest are cases when, at some point of the domain G that is a solution of equation (2), the trace of the adjugate matrix $C(\lambda, p_0, p_1)$ vanishes. In these cases we proceed analogously to the way this was done in the corresponding cases in [4].

We also note that the case of constant matrices reduces to the method under consideration indicated in [1].

2°. **Example.** Suppose it is required to find the eigenvalues and the corresponding eigenvectors of the matrix

$$A(\lambda) = \left\| \begin{matrix} 4\lambda & 3\lambda^2 + 4\lambda + 5 & 2\lambda^2 - 8\lambda + 6 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix} \right\| \tag{9}$$

for the following values of the parameter λ: 0; 0.1; 0.2; ..., 1.

For $\lambda = 0$ we have, for matrix (9), the following eigenvalues: $p_{1,2}^{(0)} = -0.5 \pm 2.39791586\,i$, $\quad p_3^{(0)} = 1$.

Choosing the step of numerical integration of the system (6), (8) equal to 0.05 and using the Runge–Kutta method, we obtain, for the eigenvalues $p_{1,2}(\lambda)$ of matrix (9), the results presented in Table 1.

Table 1

λ	$p_{1,2}$	$p_{1,2}$ exact
0.1	$-0.35000017 \pm 2.4652587\,i$	$-0.35 \pm 2.4652586\,i$
0.2	$-0.20000030 \pm 2.5219042\,i$	$-0.20 \pm 2.5219040\,i$
0.3	$-0.05000039 \pm 2.5685601\,i$	$-0.05 \pm 2.5685599\,i$
0.4	$0.09999954 \pm 2.6057631\,i$	$0.10 \pm 2.6057628\,i$
0.5	$0.24999949 \pm 2.6339137\,i$	$0.25 \pm 2.6339134\,i$
0.6	$0.39999945 \pm 2.6533001\,i$	$0.40 \pm 2.6532998\,i$
0.7	$0.54999942 \pm 2.6641136\,i$	$0.55 \pm 2.6641134\,i$
0.8	$0.69999941 \pm 2.6664586\,i$	$0.70 \pm 2.6664583\,i$
0.9	$0.84999940 \pm 2.6603574\,i$	$0.85 \pm 2.6603571\,i$
1.0	$0.99999939 \pm 2.6457516\,i$	$1.00 \pm 2.6457513\,i$

For the components of the corresponding eigenvector $X_{1,2}$ for $\lambda = 1$, we obtain the values: $-1.24999951 \mp 0.66143992\,i$; $0.37500028 \mp 0.33071858\,i$; $0.062499827 \pm 0.16535948\,i$. The residual vector: $-0.00000441 \mp 0.00000322\,i$; $0.00000027 \pm 0.00000072\,i$; $0.00000004 \pm 0.00000018\,i$.

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REFERENCES

[1] D. F. Davidenko, DAN, **131**, No. 5, 1007 (1960).
[2] D. F. Davidenko, DAN, **131**, No. 4, 731 (1960).
[3] D. F. Davidenko, DAN, **131**, No. 3, 500 (1960).
[4] D. F. Davidenko, Ukr. Math. Journal, **5**, No. 2, 196 (1953).

Note: Figure translations are in progress. See original paper for figures.