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Abstract

Full Text

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ON EQUATIONS OF PARABOLIC TYPE IN A BANACH SPACE WITH AN UNBOUNDED VARIABLE OPERATOR WHOSE FRACTIONAL POWER HAS A CONSTANT DOMAIN OF DEFINITION***

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In this paper some results obtained in ([1,2]) are carried over to the case of equations in a Banach space.

1. Consider the problem

$$\frac{dv}{dt} + A(t)v = 0 \quad (\tau < t \leq T,\ \tau \in [0,T]), \qquad v(\tau) = v_0, \tag{1}$$

where $v(t)$ is the desired function, defined on $[\tau, T]$, with values in the Banach space E; $A(t)$ $(0 \leq t \leq T)$ is a linear operator acting in E; dv/dt is the derivative, understood as the limit in the norm of E of the corresponding difference quotient.

Suppose that for each $t \in [0,T]$ the operator $A(t)$ has a domain of definition $D[A(t)]$ everywhere dense in E. Suppose that for any λ with $\operatorname{Re}\lambda \geq 0$ the operator $A(t) + \lambda I$ has a bounded inverse whose norm satisfies the inequality

$$\left\|[A(t) + \lambda I]^{-1}\right\| \leq C[|\lambda| + 1]^{-1}. \tag{2}$$

Then ([3]) the fractional powers of the operator $A(t)$ are defined.

Let ρ be some number in $(0,1)$, and let l be an integer such that $\rho_1 = 1 - l\rho \in (0, \rho]$. Suppose that the operator $A^{\rho}(t)A^{-\rho}(\tau)$ is bounded for all $t, \tau \in [0,T]$, and that the operator $A^{-\rho_1}(t)A^{\rho_1}(\tau)$ admits closure to a bounded operator. Suppose that

$$\|\Delta[A(t), A(\tau)]\| \leq C|t - \tau|^{1-\rho+\varepsilon}, \tag{3}$$

where $C > 0$, $\varepsilon \in (0, \rho]$ are some numbers, and by $\Delta[...]$ is denoted each of the bounded operators $A^\rho(t)A^{-\rho}(\tau) - I$, $A^\rho(t)A^{-\rho}(\tau) - \overline{A^{-\rho_1}(t)A^{\rho_1}(\tau)}$**. In the case $\rho \geq 1/2$, by $\Delta[...]$ is denoted the second of these operators.

Theorem 1. There exists an operator-function $U(t,\tau)$, defined for all $0 \leq \tau \leq t \leq T$, with values in the space of bounded linear operators on E, possessing the following properties:

1) $U(t,\tau)$ is uniformly continuous jointly in t and τ for all $t > \tau$, and for $t = \tau$ is strongly continuous.

2) $U(t,t) = I$, and for any $0 \leq \tau \leq s \leq t \leq T$ the identity holds

$$U(t,\tau) = U(t,s)\,U(s,\tau). \tag{4}$$

* The results of the present work were reported at the seminar on functional analysis of Voronezh State University in January 1959.

** A bar above denotes the closure of an operator in E.

3) $U(t,\tau)$, for $t > \tau$, is uniformly and continuously differentiable with respect to t, and

$$\frac{\partial U(t,\tau)}{\partial t} + A(t)U(t,\tau) = 0. \tag{5}$$

4) Problem (1), for any $v_0 \in E$, has a unique solution

$$v_\tau(t) = U(t,\tau)v_0. \tag{6}$$

continuous for all $t \geq \tau$ and continuously differentiable for $t > \tau$.

If $v_0 \in D[A(\tau)]$, then the vector-function $v_\tau(t)$ is continuously differentiable also at $t = \tau$ and satisfies equation (1). Here the derivative at the point $t = \tau$ is understood as the right derivative.

5) For any $0 \leq \tau \leq t \leq t + \Delta t \leq T$ and $\xi \in [0,T]$ the estimates

$$\|A^\alpha(t)U(t,\tau)A^{-\beta}(\tau)\| \leq C(\alpha_0)|t-\tau|^{\beta-\alpha} \quad (0 \leq \beta \leq \alpha \leq \alpha_0 < 1+\varepsilon); \tag{7}$$

$$\|A^\alpha(\xi)[U(t+\Delta t,\tau) - U(t,\tau)]A^{-\beta}(\tau)\| \leq C(\alpha,\beta,\gamma)\Delta t^{\gamma-\alpha}|t-\tau|^{\beta-\gamma}$$
$$(0 \leq \alpha \leq \rho,\ 0 \leq \beta \leq \gamma < 1+\varepsilon,\ 0 < \gamma - \alpha < 1). \tag{8}$$

In the last estimate one may have $\gamma - \alpha = 1$, if either $\xi = t$, or $\beta < \gamma$.

6) If $\varepsilon > \rho - \rho_1$, then $U(t,\tau)$ is uniformly and continuously differentiable with respect to τ for $\tau < t$, and

$$\frac{\partial U(t,\tau)}{\partial \tau} - U(t,\tau)A(\tau) = 0. \qquad (9)$$

For any $0 \le \beta \le \alpha \le \alpha_0 < 1 + \varepsilon - \rho + \rho_1$ the estimate

$$\|A^{-\beta}(t)U(t,\tau)A^{\alpha}(\tau)\| \le C(\alpha_0)|t-\tau|^{\beta-\alpha} \qquad (10)$$

is valid.

7) The operator $U(t,\tau)$ can be represented in the form of a multiplicative integral.

2. Consider the problem

$$\frac{dv}{dt} + A(t)v = f(t) \quad (0 < t \le T), \qquad v(0) = v_0. \qquad (11)$$

If the vector-function $f(t)$ is continuous, then problem (11) cannot have more than one solution continuous for all $t \ge 0$ and continuously differentiable for $t > 0$.

Theorem 2. Suppose the vector-function $f(t)$ satisfies the condition

$$\|f(t) - f(\tau)\| \le C|t-\tau|^{\delta} \quad (t,\tau \in [0,T],\ C > 0,\ 0 < \delta < 1). \qquad (12)$$

Then the formula

$$v(t) = U(t,0)v_0 + \int_0^t U(t,\tau)f(\tau)\,d\tau \qquad (13)$$

for any $v_0 \in E$ defines the unique solution of problem (11), continuous for all $t \ge 0$ and continuously differentiable for $t > 0$. If $v_0 \in D[A(0)]$, then the vector-function $v(t)$ is continuously differentiable also at $t = 0$ and satisfies equation (11). Here the derivative at the point $t = 0$ is understood as the right derivative.

3. Consider the problem

$$\frac{dv}{dt} + A(t,v)v = f(t,v) \quad (0 < t \le t_0,\ t_0 \in (0,T]), \qquad v(0) = v_0. \qquad (14)$$

Theorem 3. Let E_1 be some Banach space; let E_2 be a subset of E_1 that is also a Banach space. Suppose $\|v\|_{E_1} \le C\|v\|_{E_2}$ for any $v \in E_2$.

Suppose that for every such $v \in E_2$ that $\|v\|_{E_1} \le R_1$, $\|v\|_{E_2} \le R_2$, where R_1 and R_2 are some positive numbers, and every $t \in [0,T]$, a linear operator $A(t,v)$ acting in E, with everywhere dense domain of definition, is defined. Suppose that for every λ with $\operatorname{Re}\lambda \geq 0$ the operator $A(t,v)+\lambda I$ has a bounded inverse, and suppose that

$$\left\|[A(t,v)+\lambda I]^{-1}\right\| \leq C[|\lambda|+1]^{-1}. \tag{15}$$

Suppose that, for some $\rho \in (0,1)$, for all such $v, w \in E_2$ that $\|v\|_{E_1}, \|w\|_{E_1} \leq R_1$, $\|v\|_{E_2}, \|w\|_{E_2} \leq R_2$, and for all $t, \tau \in [0,T]$, the operator $A^{\rho}(t,v)A^{-\rho}(\tau,w)$ is bounded, and the operator $A^{-\rho_1}(t,v)A^{\rho_1}(\tau,w)$ admits closure to a bounded operator. Suppose that

$$\left\|\Delta[A(t,v),A(\tau,w)]\right\| \leq C\left[|t-\tau|^{\mu}+\|v-w\|_{E_1}^{\nu}\right], \tag{16}$$

where μ, ν are some numbers respectively from $(1-\rho,1]$, $\left(\frac{1-\rho}{\beta-\alpha},1\right]$; α, β are some numbers respectively from $[0,\rho)$, $(1-\rho+\alpha,1)$. Suppose

$$v_0 \in E_2, \quad \|v_0\|_{E_1} < R_1, \quad \|v_0\|_{E_2} < R_2.$$

Suppose that $v_0 \in D(A_0^{\beta})$, where $A_0 = A(0,v_0)$. Suppose that each element v of $D(A_0^{\alpha})$ belongs to E_1, and suppose that

$$\|v\|_{E_1} \leq C\|A_0^{\alpha}v\|. \tag{17}$$

Suppose that for any such $v, w \in E_2$ that $\|v\|_{E_2}, \|w\|_{E_2} \leq R_2$, and for all $t, \tau \in [0,T]$,

$$\|f(t,v)-f(\tau,w)\| \leq C\left[|t-\tau|^{\eta}+\|v-w\|_{E_2}^{\xi}\right], \tag{18}$$

where η and ξ are some numbers from $(0,1]$.

Suppose that E_3 and E_4 are some Banach spaces. Suppose that each element v of E_3 belongs to E_2 and E_4, and suppose that

$$\|v\|_{E_2} \leq C\|v\|_{E_3}^{\gamma}\|v\|_{E_4}^{1-\gamma}, \tag{19}$$

where γ is some number from $[0,1)$.

Suppose that each element v of $D(A_0^{\varepsilon})$, where ε is some number from $[0,\rho)$, belongs to E_4, and suppose that

$$\|v\|_{E_4} \leq C\|A_0^{\varepsilon}v\|. \tag{20}$$

Suppose that each element v of $D[A^\beta(t,w)]$, where w is any such element of E_2 that $\|w\|_{E_1} \leq R_1$, $\|w\|_{E_2} \leq R_2$, and t is any number from $[0,T]$, belongs to E_3, and suppose that

$$\|v\|_{E_3} \leq C\|A^\beta(t,w)v\|. \tag{21}$$

Finally, suppose that every set bounded in E_3 is compact in E_2. Then, for some t_0, there exists at least one solution $v(t)$ of problem (14), continuous for all $t \geq 0$ and continuously differentiable for $t > 0$.

If $E_2 = E_1$ and $\gamma = \xi = 1$, then such a solution will be unique and can be found by the method of successive approximations.

4. Suppose that Ω is an open bounded domain of n-dimensional space with boundary S. With the aid of Theorem 3, an existence theorem is proved

of a classical solution of the boundary-value problem

$$\frac{\partial v}{\partial t} - \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ik}(t)x,v)\frac{\partial v}{\partial x_k}\right] = f\left(t,x,v;\frac{\partial v}{\partial x_1},\dots,\frac{\partial v}{\partial x_n}\right) \quad (0 < t \leqslant t_0,\ x \in \Omega),$$

$$\sum_{i,k=1}^{n} a_{ik}(t,y,v)\frac{\partial v}{\partial y_k}\cos(N_y,y_i) + \sigma(t,y,v)v = 0 \quad (0 < t \leqslant t_0,\ y \in S); \tag{22}$$

$$v(0,x) = v_0(x) \quad (x \in \overline{\Omega})$$

without any restrictions on the growth of the nonlinearities. Here N_y is the vector of the exterior normal to the surface S at the point y.

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References

1. P. E. Sobolevskii, DAN, **123**, No. 6 (1958).

2. P. E. Sobolevskii, DAN, **130**, No. 2 (1960).

3. M. A. Krasnosel' skii, P. E. Sobolevskii, DAN, **129**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.