ON SOME INEQUALITIES FOR AN ENTIRE FUNCTION OF FINITE DEGREE AND ITS DERIVATIVES*

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Abstract

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ON SOME INEQUALITIES FOR AN ENTIRE FUNCTION OF FINITE DEGREE AND ITS DERIVATIVES*

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S. N. Bernstein ([1], p. 269) proved that in the class B_ν of entire functions of degree $\leqslant \nu$, bounded on the whole real axis, the following inequality holds:

$$\sup_{-\infty<x<\infty} |f'(x)| \leqslant \nu \sup_{-\infty<x<\infty} |f(x)|. \tag{1}$$

Among the numerous generalizations of this inequality we cite the following (see [2], p. 154):

If $f(z) \in B_\nu$, then for any real α,

$$\sup_{-\infty<x<\infty} |f'(x)\sin\alpha + \nu f(x)\cos\alpha| \leqslant \nu \sup_{-\infty<x<\infty} |f(x)|. \tag{2}$$

Let us note that inequality (2) is sharp for every real α.

Denote by $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(p \geqslant 1)$ the class of entire functions $f(z_1,\dots,z_n)$ of degree not exceeding $\nu_1,\dots,\nu_n$, for which the condition

$$\left(\|f\|_p^{(n)}\right)^p = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(x_1,\dots,x_n)|^p \, dx_1 \cdots dx_n < +\infty$$

is satisfied for $1 \leqslant p < +\infty$, and

$$\|f\|_\infty^{(n)} = \sup_{-\infty<x_1,\dots,x_n<+\infty} |f(x_1,\dots,x_n)| < +\infty$$

for $p = +\infty$.

For an entire function $f(z_1,\dots,z_n)$ from the class $W^{(p)}_{\nu_1,\dots,\nu_n}$ $(1 \leqslant p < p' \leqslant \infty)$, the inequality of S. M. Nikol' skii [3] holds:

$$\|f\|_{p'}^{(n)} \leqslant 2^n \prod_{k=1}^{n} \nu_k^{1/p-1/p'} \|f\|_p^{(n)}. \tag{3}$$

Inequality (3) was sharpened in the works [4-6], and it was proved that for an entire function $f(z_1, \dots, z_n) \in W^{(p)}_{\nu_1,\dots,\nu_n}$ $(1 \leqslant p < p' \leqslant \infty)$,

$$\|f\|_{p'}^{(n)} \leqslant \begin{cases} \prod_{k=1}^{n} \left(\frac{\nu_k}{\pi}\right)^{1/p-1/p'} \|f\|_p^{(n)}, & (0 < p \leqslant 2), \\ \prod_{k=1}^{n} \left(\frac{p\nu_k}{\pi}\right)^{1/p-1/p'} \|f\|_p^{(n)}, & (p > 2). \end{cases} \tag{4}$$

* The results of this note were reported at the Fifth All-Union Conference on the Theory of Functions of a Complex Variable in Yerevan in September 1960.

Inequality (4) is further refined by us as follows:

Theorem 1. Let $f(z_1, \dots, z_n) \in W^{(p)}_{\nu_1,\dots,\nu_n}$ $(1 \leq p < p' \leq \infty)$;

$$B_q = \left(\int_0^\infty \left|\frac{\sin t}{t}\right|^q dt\right)^{1/q}; \qquad B_\infty = \max_{-\infty<t<\infty} \left|\frac{\sin t}{t}\right| = 1;$$

s is the least integer not smaller than $p/2$, and the number q is chosen from the condition $1/q + s/p = 1$.

Then the inequality holds

$$\|f\|_{p'}^{(n)} \leq \left(2^{1/q}\pi^{-1}B_q\right)^{\frac{n}{s}\left(1-\frac{p}{p'}\right)} s^{n\left(\frac{1}{p}-\frac{1}{p'}\right)} \prod_{k=1}^{n} \nu_k^{\frac{1}{p}-\frac{1}{p'}} \|f\|_p^{(n)}. \tag{5}$$

In particular, taking into account that $B_q^q \leq B_2^2 = \pi/2$ $(q \geq 2)$, from (5) we find

$$\|f\|_{p'}^{(n)} \leq \prod_{k=1}^{n} \left(\frac{s\nu_k}{\pi}\right)^{1/p-1/p'} \|f\|_p^{(n)}. \tag{6}$$

Inequality (6) in the one-dimensional case was obtained independently of us by A. F. Timan (see (7), p. 248, inequality (29)).

From inequalities (2) and (5), for $p' = \infty$ it follows immediately that, for an entire function $f(z) \in W_\nu^{(p)}$ $(p \geq 1)$, the inequality holds

$$\sup_{-\infty<x<\infty} |af'(x) + b\nu f(x)| \leq \nu\sqrt{a^2+b^2}\,\left(2^{1/q}\pi^{-1}B_q\right)^{1/s} s^{1/p}\nu^{1/p}\|f\|_p. \tag{7}$$

In the case $p = 2$, inequality (7) is exact only when $a = 0$.

The present note is devoted to the refinement and generalization of inequality (7).

Introduce the notation:

$$(\alpha, \beta) = \begin{cases} \alpha - \beta, & \text{if } \alpha > \beta, \\ 0, & \text{if } \alpha \le \beta, \end{cases}$$

$$D_\nu[f; z; a, b; \alpha', \alpha''] = a \prod_{k=1}^{n} \nu_k^{(\alpha_k'', \alpha_k')} \frac{\partial^{\alpha_1' + \dots + \alpha_n'} f(z)}{\partial x_1^{\alpha_1'} \cdots \partial x_n^{\alpha_n'}} +$$

$$+ b \prod_{k=1}^{n} \nu_k^{(\alpha_k', \alpha_k'')} \frac{\partial^{\alpha_1'' + \dots + \alpha_n''} f(z)}{\partial x_1^{\alpha_1''} \cdots \partial x_n^{\alpha_n''}},$$

where $z = (z_1, \dots, z_n)$; a and b are complex numbers; α_k' and α_k'' $(k = 1, 2, \dots, n)$ are nonnegative integers.

Theorem 2. For an entire function $f(z_1, \dots, z_n) \in W^{(p)}_{\nu_1, \dots, \nu_n}$ $(1 \le p \le 2)$, we have

$$|D_\nu[f; z; a, b; \alpha', \alpha'']| \le \frac{C_q}{\pi^n} \prod_{k=1}^{n} \nu_k^{\max(\alpha_k', \alpha_k'') + 1/p} \|f\|_p^{(n)}, \tag{8}$$

where

$$C_q = \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| a \prod_{k=1}^{n} \frac{\partial^{\alpha_k'}}{\partial x_k^{\alpha_k'}} \left(\frac{\sin(x_k + i\nu_k y_k)}{x_k + i\nu_k y_k} \right) + \right.\right.$$

$$\left.\left. + b \prod_{k=1}^{n} \frac{\partial^{\alpha_k''}}{\partial x_k^{\alpha_k''}} \left(\frac{\sin(x_k + i\nu_k y_k)}{x_k + i\nu_k y_k} \right) \right|^q dx_1 \cdots dx_n \right\}^{1/q}$$

and the number q is determined from the condition $1/p + 1/q = 1$.

It is obvious that, when $y_k = 0$ $(k = 1, 2, \dots, n)$, $p = 2$, and the numbers a and b are real, if for at least one value of k the numbers α_k' and α_k'' have different parity, inequality (8) takes the form:

$$|D_\nu\{f; x; a, b; \alpha', \alpha''\}| \le$$

$$\leq \left[a^2 \prod_{k=1}^{n} \frac{1}{2\alpha'_k + 1} + b^2 \prod_{k=1}^{n} \frac{1}{2\alpha''_k + 1}\right]^{-1/2} \frac{1}{\pi^{n/2}} \prod_{k=1}^{n} \nu_k^{\max(\alpha'_k, \alpha''_k)+1/2} \|f\|_2^{(n)},$$

where $x = (x_1, \dots, x_n)$.

The last inequality becomes an equality for $x_1 = \dots = x_n = 0$ for the function

$$f_0(x_1, \dots, x_n) = a \prod_{k=1}^{n} \nu_k^{(\alpha''_k, \alpha'_k)} \left(\frac{\sin \nu_k x_k}{x_k}\right)^{(\alpha'_k)} + b \prod_{k=1}^{n} \nu_k^{(\alpha'_k, \alpha''_k)} \left(\frac{\sin \nu_k x_k}{x_k}\right)^{(\alpha''_k)},$$

which is an entire function from the class $W^{(2)}_{\nu_1, \dots, \nu_n}$.

In particular, let $b = 0$, $1 \leq p \leq 2$, $y_k = 0$ $(k = 1, 2, \dots, n)$, and let $\alpha'_k = \alpha_k$ be arbitrary nonnegative numbers, while $\alpha''_k = 0$ $(k = 1, 2, \dots, n)$. In this case, for an entire function $f(z_1, \dots, z_n) \in W^{(p)}_{\nu_1, \dots, \nu_n}$ $(1 \leq p \leq 2)$, we have

$$\left|\frac{\partial^{\alpha_1 + \dots + \alpha_n} f(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right| \leq \frac{A_q}{\pi^n} \prod_{k=1}^{n} \nu_k^{\alpha_k + 1/p} \|f\|_p^{(n)}, \tag{9}$$

where

$$A_q = \prod_{k=1}^{n} \left\| \left(\frac{\sin x}{x}\right)^{(\alpha_k)} \right\|_q.$$

For $p = 2$, inequality (9) becomes an equality for the function

$$\prod_{k=1}^{n} \left(\frac{\sin kx}{kx}\right)^{(\alpha_k)} \quad \text{when } x_1 = \dots = x_n = 0.$$

From inequality (9) one obtains the inequality that was previously obtained in papers ([5],[6]):

$$\left|\frac{\partial^{\alpha_1 + \dots + \alpha_n} f(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right| \leq \prod_{k=1}^{n} \nu_k^{\alpha_k} \left(\frac{\nu_k}{\pi}\right)^{1/p} (p\alpha_k + 1)^{-1/p} \|f\|_p^{(n)}.$$

Theorem 3. Let s be the smallest integer not less than $p/2$ $(p \geq 1)$, let the number q be chosen from the condition $1/q + s/p = 1$, and let y_k be arbitrary real numbers. Then, for an entire function $f(z_1, \dots, z_n) \in W^{(p)}_{\nu_1, \dots, \nu_n}$ $(p \geq 1)$, the inequality

$$|f(x_1+iy_1,\dots,x_n+iy_n)| \le \prod_{k=1}^{n}\left(\frac{s\nu_k}{\pi}\right)^{1/p}\left[\frac{\operatorname{sh}(p\nu_k y_k)}{p\nu_k y_k}\right]^{1/p}\|f\|_p^{(n)} \qquad (10)$$

holds.

Inequality (10) is a refinement of the corresponding inequality obtained in paper ([8]).

Let $\varphi(x_1,\dots,x_n)\ge 1$ be a fixed function, continuous

in the n-dimensional Euclidean space R_n:

$$\alpha(t_1,\dots,t_n)=\sup_{\substack{-\infty<x_1,\dots,x_n<\infty\\ |y_1|\le t_1,\dots,|y_n|\le t_n}}\frac{\varphi(x_1+y_1,\dots,x_n+y_n)}{\varphi(x_1,\dots,x_n)}\le$$

$$\le\sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n}A_{k_1,\dots,k_n}t_1^{k_1}\cdots t_n^{k_n}\equiv M(t_1,\dots,t_n).$$

Denote by $W^{(p;\varphi)}_{\nu_1,\dots,\nu_n}$ the class of entire functions $f(x_1,\dots,x_n)$ of degree $\nu_1,\dots,\nu_n$, for which the following Lebesgue integral is finite:

$$(\|f\|^{(n)}_{p,\varphi})^p=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\frac{f(x_1,\dots,x_n)}{\varphi(x_1,\dots,x_n)}\right|^p dx_1\cdots dx_n.$$

Theorem 4. If a,b are real numbers, each of the numbers α'_k, α''_k $(k=1,2,\dots,n)$ independently of the others assumes the values zero and one, and, moreover, for at least one value of k the numbers α'_k and α''_k are not equal, then

$$\frac{|D_{\nu+\lambda}[f;x;a,b;\alpha',\alpha'']|}{\varphi(x_1,\dots,x_n)}\le$$

$$\le\left[a^2\prod_{k=1}^{n}\frac{1}{2\alpha'_k+1}+b^2\prod_{k=1}^{n}\frac{1}{2\alpha''_k+1}\right]^{1/2}\times$$

$$\times\pi^{-n/2}\prod_{k=1}^{n}(\nu_k+\lambda_k)^{\max(\alpha'_k,\alpha''_k)+1/2}M\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_n}\right)\|f\|^{(n)}_{2,n},$$

where λ_k are arbitrary positive parameters; $D_{\nu+\lambda}[f;x;a,b;\alpha',\alpha'']$ is obtained from the expression $D_\nu[f;x;a,b;\alpha',\alpha'']$ by replacing ν_k by $\nu_k+\lambda_k$ $(k=1,2,\dots,n)$. In the case $\varphi(x_1,\dots,x_n)=\prod_{k=1}^{n}(1+x_k^2)$, $\lambda_k=\sqrt{\nu_k}$, and as $\nu_k\to\infty$, the last inequality turns into an asymptotic equality at $x_1=x_2=\dots=x_n=0$

for the function $g_0(x_1, \dots, x_n) = \varphi(x_1, \dots, x_n) f_0(x_1, \dots, x_n)$, which is an entire function of degree $\nu_1, \dots, \nu_n$, where $f_0(x_1, \dots, x_n)$ is defined above.

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Note: Figure translations are in progress. See original paper for figures.