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Abstract

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ON AN EXACT METHOD FOR COMPUTING CERTAIN INTEGRALS WITH A CAUCHY-TYPE KERNEL

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The solution of many problems in continuum mechanics reduces to the computation of singular integrals with a Cauchy-type kernel, taken along a segment of the real axis. In the case where the kernel is singular, these integrals may be reduced to either of two integrals of the following form:

$$J(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt \qquad (-1 \le x \le 1); \tag{1}$$

$$I(x) = \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} \qquad (-1 \le x \le 1). \tag{2}$$

In the case of a regular kernel, they may be reduced to either of two integrals of the form

$$G(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt \qquad (1 \le x \le \infty); \tag{3}$$

$$E(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} \qquad (1 \le x \le \infty). \tag{4}$$

The integrals $J(x)$ and $I(x)$ exist in the sense of the Cauchy principal value, and for their existence in this sense it is sufficient to require that the function $f(x)$ satisfy the Hölder condition ([1,2]) on the interval $[-1, 1]$. The integrals $G(x)$ and $E(x)$ for $x > 1$ are ordinary Riemann integrals. At the point $x = 1$ the integral $G(x)$ may have a logarithmic-type singularity. At present there are several works ([3–6]) in which various methods are given for computing the integrals under consideration, allowing, under certain restrictions on the integrand $f(x)$, the latter to be computed with a prescribed degree of accuracy. However, there exists a large class of functions $f(x)$ for which the integrals (1)–(4) can be expressed through elementary functions or through certain known, well-tabulated functions. Examples of such functions are rational functions. Identifying all possible cases in which the integrals (1)–(4) can be represented in closed form through known functions is, in our opinion, of practical interest, since these integrals occur (as noted above) in solving many problems of continuum mechanics. In this connection, the present note gives a method that makes it possible to express the integrals (1)–(4) through elementary functions or through certain known tabulated functions for a fairly broad class of functions $f(x)$.

1. The essence of the proposed method consists in the following proposition:

Theorem. Let $F(\zeta)$ be a function of the complex variable $\zeta = \xi + i\eta$, analytic in the circle $|\zeta| \leqslant 1$ and satisfying the condition

$$\operatorname{Im} F(\zeta) = 0 \quad \text{when } \eta = 0; \tag{5}$$

then the following equalities hold:

$$J(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\operatorname{Im} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\,dt = \operatorname{Re} F\left(x - i\sqrt{1-x^2}\right) - F(0)$$

$$(-1 \leqslant x \leqslant 1); \tag{6}$$

$$I(x) = \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1} \frac{\operatorname{Re} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\,\frac{dt}{\sqrt{1-t^2}} = -\operatorname{Im} F\left(x - i\sqrt{1-x^2}\right)$$

$$(-1 \leqslant x \leqslant 1); \tag{7}$$

$$G(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\operatorname{Im} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\,dt = F\left(x - \sqrt{x^2-1}\right) - F(0) \quad (1 \leqslant x \leqslant \infty); \tag{8}$$

$$E(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1} \frac{\operatorname{Re} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\,\frac{dt}{\sqrt{1-t^2}} = -F\left(x - \sqrt{x^2-1}\right)$$

$$(1 \leqslant x \leqslant \infty). \tag{9}$$

Proof. In view of condition (5), the analytic function $F(\zeta)$ is represented in the circle $|\zeta| \leqslant 1$ in the form of the series

$$F(\zeta) = F(0) + \sum_{n=1}^{\infty} a_n \zeta^n, \tag{10}$$

where $F(0), a_1, \dots, a_n, \dots$ are real numbers.

Putting in equality (10) $\zeta = x - i\sqrt{1-x^2}$ $(-1 \leqslant x \leqslant 1)$ and taking into account that

$$\left(x - i\sqrt{1-x^2}\right)^n = T_n(x) - iU_n(x)$$

$(-1 \leqslant x \leqslant 1)$, where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, determined by the equalities

$$T_n(x) = \cos n \arccos x, \qquad U_n(x) = \sin n \arccos x,$$

we shall have

$$\operatorname{Im} F\left(x - i\sqrt{1-x^2}\right) = -\sum_{n=1}^{\infty} a_n U_n(x) \quad (-1 \leqslant x \leqslant 1); \tag{11}$$

$$\operatorname{Re} F\left(x - i\sqrt{1-x^2}\right) = F(0) + \sum_{n=1}^{\infty} a_n T_n(x) \quad (-1 \leqslant x \leqslant 1). \tag{12}$$

Putting in equality (10) $\zeta = x - \sqrt{x^2-1}$ $(1 \leqslant x \leqslant \infty)$, we obtain

$$F\left(x - \sqrt{x^2-1}\right) = F(0) + \sum_{n=1}^{\infty} a_n \left(x - \sqrt{x^2-1}\right)^n \quad (1 \leqslant x \leqslant \infty). \tag{13}$$

Substitute, in the right-hand sides of equalities (1), (3) and (2), (4), instead of the function $f(x)$, the expansions (11) and (12), respectively, and then use the relations

$$\frac{1}{\pi}\int_{-1}^{1} \frac{U_n(t)}{t-x}\,dt = -T_n(x), \qquad \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1} \frac{T_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} = U_n(x) \quad (-1 \leqslant x \leqslant 1),$$

$$\frac{1}{\pi}\frac{U_n(t)}{t-x}\,dt = -\left(x - \sqrt{x^2-1}\right)^n,$$

$$\frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1} \frac{T_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} = -\left(x - \sqrt{x^2-1}\right)^n \quad (1 \leqslant x \leqslant \infty),$$

the validity of which is easy to prove by using the method of mathematical induction or by means of the Sokhotskii formulas ([1,2]); as a result we find that

$$J(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\operatorname{Im} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\, dt = \sum_{n=1}^{\infty} a_{nT} n(x) \qquad (-1 \le x \le 1);$$

$$I(x) = \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1} \frac{\operatorname{Re} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\, \frac{dt}{\sqrt{1-t^2}} = \sum_{n=1}^{\infty} a_{nU} n(x)$$

$$(-1 \le x \le 1);$$

$$G(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\operatorname{Im} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\, dt = \sum_{n=1}^{\infty} a_n \left(x - \sqrt{x^2-1}\right)^n \qquad (1 \le x \le \infty);$$

$$E(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1} \frac{\operatorname{Re} F\left(t - i\sqrt{1-t^2}\right)}{t-x}\, \frac{dt}{\sqrt{1-t^2}} =$$

$$= -F(0) - \sum_{n=1}^{\infty} a_n \left(x - \sqrt{x^2-1}\right)^n \qquad (1 \le x \le \infty).$$

Substituting now into the right-hand sides of the last equalities, in place of the series standing there, their values according to (11)–(13), we obtain the desired formulas (6)–(9). The formulas obtained, (6)–(9), give a method for determining the exact value of the integrals $J(x)$ and $G(x)$ for functions $f(x)$ satisfying the condition $f(x) = \operatorname{Im} F\left(x - i\sqrt{1-x^2}\right)$, and of the integrals $I(x)$ and $E(x)$ for functions $f(x)$ satisfying the condition $f(x) = \operatorname{Re} F\left(x - i\sqrt{1-x^2}\right)$, where $F(\zeta)$ is the analytic function defined above.

2. Let us illustrate the method described by examples.

1) Consider the function

$$F(\zeta) = \arctan\left(\zeta \tan\frac{\alpha}{2}\right),$$

which is analytic in the disk $|\zeta| \le 1$ and satisfies condition (5). In the present case

$$\operatorname{Im} F\left(x - i\sqrt{1-x^2}\right) = \frac{1}{4}\ln\frac{1-\sqrt{1-x^2}\sin\alpha}{1+\sqrt{1-x^2}\sin\alpha} \qquad (-1 \le x \le 1);$$

$$\operatorname{Re} F\left(x - i\sqrt{1-x^2}\right) = \frac{1}{2}\arctan(x\tan\alpha) \qquad (-1 \le x \le 1);$$

$$F\left(x-\sqrt{x^2-1}\right)=\arctan\left(\left(x-\sqrt{x^2-1}\right)\tan\frac{\alpha}{2}\right)\qquad(1\le x\le\infty).$$

Substituting the expressions found into formulas (6)–(9) and taking into account that $F(0)=0$, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{t-x}\ln\frac{1-\sqrt{1-t^2}\sin\alpha}{1+\sqrt{1-t^2}\sin\alpha}\,dt=2\arctan(x\tan\alpha)\qquad(-1\le x\le 1);$$

$$\frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{\arctan(t\tan\alpha)}{t-x}\frac{dt}{\sqrt{1-t^2}}=-\frac{1}{2}\ln\frac{1-\sqrt{1-x^2}\sin\alpha}{1+\sqrt{1-x^2}\sin\alpha}\qquad(-1\le x\le 1);\tag{14}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{t-x}\ln\frac{1-\sqrt{1-t^2}\sin\alpha}{1+\sqrt{1-t^2}\sin\alpha}\,dt=4\arctan\left(\left(x-\sqrt{x^2-1}\right)\tan\frac{\alpha}{2}\right),$$

$$(1\le x\le\infty);$$

$$\frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{\arctan(t\tan\alpha)}{t-x}\frac{dt}{\sqrt{1-t^2}}=-2\arctan\left(\left(x-\sqrt{x^2-1}\right)\tan\frac{\alpha}{2}\right)$$

$$(1\le x\le\infty).$$

2) Consider another function

$$F(\zeta)=\frac{1}{\pi}\left(\zeta-\frac{1}{\zeta}\right)\int_0^{\zeta}\ln\frac{1+t}{1-t}\frac{dt}{t}.$$

It is easy to see that this function is analytic in the disk $|\zeta|\leqslant 1$, satisfies condition (5), and at the point $\zeta=0$ takes the value $F(0)=-2/\pi$. After simple transformations we find

$$\operatorname{Im}F\left(x-i\sqrt{1-x^2}\right)=-\sqrt{1-x^2}\arcsin x\quad(-1\leqslant x\leqslant 1);$$

$$\operatorname{Re}F\left(x-i\sqrt{1-x^2}\right)=\frac{4}{\pi}\sqrt{1-x^2}\left[L(u(x))+L\left(\frac{\pi}{2}-u(x)\right)-L\left(\frac{\pi}{2}\right)\right],$$

$$u(x)=\frac{1}{2}\arccos x \quad (-1\leqslant x\leqslant 1);$$

$$F\left(x-\sqrt{x^2-1}\right)=-\frac{2}{\pi}N(v(x))\sqrt{x^2-1};$$

$$v(x)=\frac{1}{x+\sqrt{x^2-1}} \quad (1\leqslant x\leqslant \infty).$$

$L(u)$ is Lobachevskii' s function,

$$L(u)=-\int_0^u \ln\cos\varphi\, d\varphi \quad \left(0\leqslant u\leqslant \frac{\pi}{2}\right),$$

and $N(v)$ is defined by the equality

$$N(v)=\int_0^v \ln\frac{1+t}{1-t}\,\frac{dt}{t} \quad (0\leqslant v\leqslant 1).$$

After substituting the expressions found above into formulas (6)–(9), we obtain the exact values of the following four integrals:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\arcsin t}{t-x}\,dt=\frac{4}{\pi}\sqrt{1-x^2}\left[L\left(\frac{\pi}{2}\right)-L(u(x))-L\left(\frac{\pi}{2}-u(x)\right)\right]-\frac{2}{\pi} \quad (-1\leqslant x\leqslant 1),$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{L(u(t))+L\left(\frac{\pi}{2}-u(t)\right)-L\left(\frac{\pi}{2}\right)}{t-x}\,dt=\frac{\pi}{4}\arcsin x \quad (-1\leqslant x\leqslant 1);$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\arcsin t}{t-x}\,dt=\frac{2}{\pi}\sqrt{x^2-1}\,N(v(x))-\frac{2}{\pi} \quad (1\leqslant x\leqslant \infty);$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{L(u(t))+L\left(\frac{\pi}{2}-u(t)\right)-L\left(\frac{\pi}{2}\right)}{t-x}\,dt=\frac{1}{2}N(v(x)) \quad (1\leqslant x\leqslant \infty). \qquad (15)$$

Let us note that the integrals (14) occur in solving the problem of jet flow around certain curvilinear obstacles ([7]), and the integrals (15) enter into the solution of the problem of the rotational impact of a plate around which a jet separates ([8]). Using formulas (6)–(9), it is not difficult to compile tables of exact values of the integrals (1)–(4), which may be useful in reducing to numerical results various problems of continuum mechanics.

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