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Abstract

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ON THE STABILITY OF SOLUTIONS OF SECOND-ORDER EQUATIONS

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1. We first give comparison theorems.

Theorem 1. Let $f(v_1, v_2, v_3)$ be nonincreasing in v_1, and let the functions $x_1(t)$, $x_2(t)$ satisfy the conditions

$$\ddot{x}_1 = f(t, x_1, \dot{x}_1) \qquad (t \geq t_1), \tag{1}$$

$$\ddot{x}_2 < f(t, x_2, \dot{x}_2) \qquad (t \geq t_2 \geq t_1), \tag{2}$$

$$x_1(t_1) = x_2(t_2), \qquad \dot{x}_1(t_1) \geq \dot{x}_2(t_2) \geq 0. \tag{3}$$

Let t_3 $(t_1 < t_3 \leq \infty)$ be the nearest point to the right of t_1 at which the function $x_1(t)$ attains a maximum, and suppose that on the interval $[t_2, t_4]$ the inequality $x_2(t) \geq x_2(t_2)$ holds. Then the inequality

$$x_2(t) < x_1(t_3) \qquad (t_2 < t \leq t_4) \tag{4}$$

holds.

If the uniqueness theorem is valid for equation (1), then Theorem 1 remains valid when the sign $<$ in (2) and (4) is replaced by $\leq$.

It is enough, obviously, to prove inequality (4) only for those t for which $\dot{x}_2(t) \geq 0$. Therefore one may assume that $x_2(t)$ is increasing on $[t_2, t_4]$ (otherwise the matter reduces to a relabeling of t_1 and t_2). We take x as the independent variable and note that $\ddot{x} = d\left({}^1\!/_2\dot{x}^2\right)/dx$. Using (1), (2), and the nonincrease of $f(v_1, v_2, v_3)$ in the first argument, we obtain the possibility of applying Chaplygin' s theorem on differential inequalities to $\dot{x}_1^2/2$, $\dot{x}_2^2/2$, regarded as functions of x. Hence it follows that if t_5 $(t_1 < t_5 \leq t_3)$ and t_6 $(t_2 < t_6 \leq t_4)$ are, respectively, the points of intersection of the graphs of the functions $x_1(t)$ and $x_2(t)$ with a straight line of the form $x = \text{const}$, then $\dot{x}_1(t_5) > \dot{x}_2(t_6)$. It remains to apply the last assertion to the line $x = x_1(t_3)$.

The proposition just proved pertains, in essence, to increasing $x_1(t)$, $x_2(t)$; by the same method it is not difficult to obtain an analogous result also for decreasing functions. We shall need below a corollary of Theorem 1, which, for brevity, we formulate here only for the case corresponding to the first quadrant of the phase plane.

Corollary. Let $-g(v_1, v_2, v_3) \geq av_2 + bv_3$ $(a > 0,\ b > -2\sqrt{a})$ for nonnegative v_2 and v_3; let the solution $x(t)$ $(x(t_0) = 0)$ of the equation $\ddot{x} = g(t, x, \dot{x})$ be nonnegative on the interval $[t_0, t_1]$. Then for $t_0 \leq t \leq t_1$ the inequality

$$x(t) \leq \begin{cases} \dfrac{1}{\sqrt{a}}\,\dot{x}(t_0)e^{-\gamma/\operatorname{tg}\gamma}, & \text{if } 1 \geq \dfrac{b}{2\sqrt{a}} = \cos\gamma,\ 0 \leq \gamma < \pi, \\ \dfrac{1}{\sqrt{a}}\,\dot{x}(t_0)e^{-\gamma/\operatorname{th}\gamma}, & \text{if } 1 < \dfrac{b}{2\sqrt{a}} = \operatorname{ch}\gamma. \end{cases} \tag{5}$$

For some particular cases, estimates analogous to (5) were known earlier. Thus, in (3) an analogue of inequality (5) is given for the linear equation $\ddot{x} + p(t)\dot{x} + q(t)x = 0$ in the case $p(t) \equiv \text{const}$. In (1) an estimate analogous to (5) is established for the equation

$$A\ddot{y} + [B + R(\dot{y})]\dot{y} + Cy = 0, \tag{6}$$

where A, B, C are constants; $A, C > 0$; $B \geqslant 0$; $R(v)$ is a nonnegative even function satisfying the condition $R(0) = 0$ (the last two properties of $R(v)$ are immaterial for this estimate). Sansone considers here only the case $B^2 - 4AC \leqslant 0$; inequality (5) immediately also yields the corresponding estimate in the case $B^2 - 4AC > 0$. There Sansone also states the following assertion (for the case $B^2 - 4AC \leqslant 0$): if t_s and t_{s+1} are adjacent zeros of a solution $y(t)$ of equation (6), and $z(t)$ is a solution of the equation $A\ddot{z} + B\dot{z} + Cz = 0$, satisfying the conditions $z(t_s) = y(t_s) = 0$, $\dot{z}(t_s) = \dot{y}(t_s) > 0$, then for $t_s < t < t_{s+1}$ the inequality $y(t) < z(t)$ holds. It is easy to see that this assertion is erroneous. It is not confirmed even in the simplest case $R(v) \equiv \text{const} > 0$; the fact that here the condition $R(0) = 0$ is not satisfied is immaterial, since one can put, for example, $R_n(v) = 1 - e^{-nv^2}$ and pass to the limit as $n \to \infty$. In reality, the validity of the inequality $y(t) < z(t)$ can be guaranteed only on an interval $(t_s, t'_s]$ of monotonicity of $z(t)$.

2. Let us turn to the question of the stability of solutions of the linear equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0, \tag{7}$$

whose coefficients, for simplicity, we shall assume to be piecewise continuous on (t_0, ∞). Since the cases of oscillatory and nonoscillatory solutions differ substantially, the most effective method of investigation appears to be to consider each of these two cases separately and to synthesize the results obtained. The following criterion shows that, when $q(t) \geqslant 0$, the principal difficulty is presented by the oscillatory case.

Theorem 2. *Let $q(t) \geqslant 0$, and let the solutions of equation (7) be nonoscillatory. Then: a) in order that all solutions of equation (7) be bounded on (t_0, ∞), it is necessary and sufficient that the condition*

$$\int^{\infty} \exp\left(-\int^{t} p(\tau)\, d\tau\right) dt < \infty; \tag{8}$$

be satisfied; b) in order that all solutions of equation (7) tend to zero as $t \to \infty$, it is necessary and sufficient that (8) hold and that the condition

$$\int^{\infty} dt \int^{t} q(s) \exp\left(-\int_{s}^{t} p(\tau)\, d\tau\right) ds = \infty. \tag{9}$$

be satisfied.

Let us note that if, in the conditions of Theorem 2, $p(t)$ is bounded below and $q(t)$ above, then the derivatives of all solutions of equation (7) tend to zero as $t \to \infty$. Less general results were obtained in (4). Thus, under the assumption

$$0 < m \leqslant q(t) \leqslant M, \qquad p(t) \geqslant l > 0 \tag{10}$$

(to exclude oscillation, in (4) the additional condition $l \geqslant 2\sqrt{M}$ is imposed), an analogue of (9) is again given,

$$\int^{\infty} dt \int^{t} \exp\left(-\int_{s}^{t} p(\tau)\, d\tau\right) ds = \infty \tag{11}$$

as a necessary and sufficient condition for all solutions of (7) to tend to zero as $t \to \infty$. The validity of (8) in this case is ensured by the second condition (10).

Theorem 3. In order that equation (7) have a fundamental system of solutions of the form $x_1(t) = o(1)$, $x_2(t) = 1 + o(1)$, it is sufficient, and in the case of sign-constant $q(t)$ also necessary, that the condition

$$\int^{\infty} dt \int^{t} |q(s)| \exp\left(-\int_{s}^{t} p(\tau)\, d\tau\right) ds < \infty. \tag{12}$$

be satisfied.

This proposition is also a generalization of Opial' s result ([4]): if $|q(t)| \le M$ and the integral on the left-hand side of (11) converges, then equation (7) has a fundamental system of solutions of the form $x_1(t) = o(1)$, $x_2(t) = 1 + o(1)$.

3. Suppose now that the coefficients of equation (7) satisfy inequalities (10), and let us determine what relations between l, m, M are sufficient conditions for stability and asymptotic stability of the solutions. Theorem 2 shows that in the nonoscillatory case stability will hold, and, when condition (11) is fulfilled, also asymptotic stability, for arbitrary l, m, M. In the oscillatory case, however, the numerical relation between l, m, M turns out to be very essential. We do not include in conditions (10) the inequality $p(t) \le L$, since the numerical value of L can be used to refine the criteria given below.

Let $x(t)$ be an oscillatory solution of equation (7), for which t_n and t_{n+1} are neighboring points of maximum modulus. For studying the asymptotics of $x(t)$, as is well known, an estimate of the ratio $|x(t_{n+1})/x(t_n)|$ is of great importance. Representing this ratio in the form $|x(t_{n+1})/x(t_n)| = |x(t_{n+1})/\dot{x}(t'_n)| : |x(t_n)/\dot{x}(t'_n)|$, where t'_n is a zero of $x(t)$ in the interval (t_n, t_{n+1}), and applying analogues of (5) for different quadrants of the phase plane, we obtain the basic estimate

$$\left|\frac{x(t_{n+1})}{x(t_n)}\right| \le \varphi(l,m,M) = \begin{cases} \dfrac{\cos\gamma_2}{\cos\gamma_1}\, e^{-\left(\dfrac{\pi-\gamma_1}{\operatorname{tg}\gamma_1} + \dfrac{\gamma_2}{\operatorname{tg}\gamma_2}\right)}, & \text{for } l \le 2\sqrt{m}, \\[2ex] \dfrac{\operatorname{ch}\gamma_2}{\cos\gamma_1}\, e^{-\left(\dfrac{\pi-\gamma_1}{\operatorname{tg}\gamma_1} + \dfrac{\gamma_2}{\operatorname{th}\gamma_2}\right)}, & \text{for } l > 2\sqrt{m}. \end{cases} \tag{13}$$

Here it is put (in view of oscillation, $l < 2\sqrt{M}$): $l/2\sqrt{M} = \cos\gamma_1$ $(0 < \gamma_1 < \pi/2)$; $l/2\sqrt{m} = \cos\gamma_2$ $(0 \le \gamma_2 < \pi/2)$ for $l \le 2\sqrt{m}$; $l/2\sqrt{m} = \operatorname{ch}\gamma_2$ for $l > 2\sqrt{m}$.

4. Put $\psi(l,m,M) = 0$ for $l \ge 2\sqrt{m}$; $\psi(l,m,M) = \varphi(l,m,M)$ for $l < 2\sqrt{M}$. Estimate (13) allows one to formulate the following criteria of stability and asymptotic stability.

Theorem 4. Let the coefficients of equation (7) satisfy conditions (10), and moreover

$$\psi(l,m,M) \le 1. \tag{14}$$

Then all solutions of equation (7), together with their derivatives, are bounded on (t_0, ∞).

Theorem 5. Let the coefficients of equation (7) satisfy conditions (10), (11), and moreover

$$\psi(l, m, M) < 1. \tag{15}$$

Then all solutions of equation (7), together with their derivatives, tend to zero as $t \to \infty$.

We note that condition (11) is fulfilled, in particular, if $p(t)$ is bounded above.

V. M. Starzhinskii (see, for example, ([2])) obtained by another method a less precise result: if $0 < l \leq p(t) \leq L$, $0 < m \leq q(t) \leq M$, then a sufficient condition for asymptotic stability for equation (7) is the fulfillment of the inequalities

$$l > \sqrt{M} - \sqrt{m}, \qquad L < \frac{M + 2\sqrt{Mm} + 5m}{\sqrt{M} - \sqrt{m}}. \tag{16}$$

Comparison of conditions (15) and (16) shows that the second of conditions (16) is superfluous, while the first may be replaced, for example, by the following:

$$l \geq \frac{2}{\pi}\left(\sqrt{M} - \sqrt{m}\right). \tag{17}$$

Condition (17), unlike (15), is not sharp (although the value of the constant $2/\pi$ cannot be improved). However, it is simpler to verify; moreover, the discrepancy between (15) and (17) is insignificant, and for m close to M, conditions (15) and (17) practically coincide, as shown by the relation

$$(\varepsilon = 1 - m/M) \quad \psi\left(\frac{2}{\pi}\left(\sqrt{M} - \sqrt{m}\right), m, M\right) = 1 - \frac{1}{48}\left(1 - \frac{9}{\pi^2}\right)\varepsilon^3 - \cdots.$$

5. Suppose now that the coefficients of equation (7) satisfy the less restrictive conditions

$$0 \leq q(t) \leq M, \qquad p(t) \geq l > 0. \tag{18}$$

A study of the limiting behavior of the function $\psi(l, m, M)$ as $m \to 0$ shows that in this case effective criteria for stability and asymptotic stability can also be established. Let h_0 ($h_0 \approx 3.046$) be the root of the equation

$$\ln h\sqrt{4h - 1} - 2\arcsin\frac{1}{2\sqrt{h}} = \pi.$$

Theorem 6. *Let the coefficients of equation (7) satisfy conditions (18), and suppose that*

$$M \leq h_0 l^2. \tag{19}$$

Then all solutions of equation (7), together with their derivatives, are bounded on (t_0, ∞).

Theorem 7. *Let the coefficients of equation (7) satisfy conditions (9), (18), and suppose that*

$$M < h_0 l^2. \tag{20}$$

Then all solutions of equation (7), together with their derivatives, tend to zero as $t \to \infty$.

We note that condition (9) is satisfied, in particular, if $q(t)$ is bounded below by a positive constant and $p(t)$ is bounded above.

6. Criteria 4-7 are sharp (L-criteria in the terminology of V. M. Starzhinskii ([2])). With respect, for example, to Theorem 7 this means the following. If l, M are arbitrary positive numbers for which condition (20) is not satisfied, then one can indicate an equation of the form (7) with coefficients satisfying conditions (9), (18), possessing solutions that do not tend to zero at infinity. On the other hand, if (9) is not satisfied, then, independently of the values of l and M in conditions (18), asymptotic stability also does not occur. This follows, for example, from Theorem 3. We note that, with the aid of estimate (13) and the substitution $x(t) = \tilde{x}(-t)$, analogous criteria of instability for equation (7) can be obtained in the case $p(t) \leq l < 0$.

In conclusion, we observe that since inequality (5) was established without the assumption of linearity, most of the results presented, with certain modifications, can be transferred to nonlinear equations

$$\ddot{x} = f(t, x, \dot{x}),$$

whose right-hand sides are majorized by functions of the form $ax + b\dot{x}$. Theorem 1 can also be applied in essentially nonlinear cases; the matter then reduces to the choice of a corresponding "model" equation.

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