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Abstract

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A CONTINUOUS ANALOG OF POLYNOMIALS ORTHOGONAL ON AN ARC OF A CIRCLE

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1. Orthogonal polynomials on a circle, in contrast to orthogonal trigonometric sums, contain only nonnegative powers
$1, z, z^2, \ldots$ of the quantity $z = e^{it}$. Accordingly, their continuous analog—a family $P(x, \lambda)$, continuous in x (≥ 0), of entire transcendental functions of finite degree in λ—is obtained by means of a "continuous" orthogonalization of the family $e^{it\lambda}$, where the parameter t ranges only over nonnegative real values. The characteristic properties of the family $P(x, \lambda)$ are as follows: a) $P(0, \lambda) = 1$; b) if $x > 0$, then $P(x, \lambda)$ is an entire function of exact degree x, bounded in the half-plane $\operatorname{Im} \lambda \geq 0$, and such that
$\lim_{|\lambda| \to \infty} P(x, \lambda) e^{-ix\lambda} = 1$ ($\operatorname{Im} \lambda \leq 0$); c) there exists a nondecreasing function $\sigma(\lambda)$ ($-\infty < \lambda < \infty$), for which

$$\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty, \tag{1}$$

and such that for every continuous finite function $f(x)$ ($x \geq 0$)

$$\int_0^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \left| \int_0^{\infty} f(x) P(x, \lambda) \, dx \right|^2 d\sigma(\lambda). \tag{2}$$

Property c) expresses the fact that the operator U in $\mathcal{L}^2(0, \infty)$, defined by the formula

$$Uf = \underset{n \to \infty}{\text{l.i.m.}} \int_0^n f(x) P(x, \lambda) \, dx = F(\lambda), \tag{3}$$

transforms $\mathcal{L}^2(0, \infty)$ into a certain linear manifold $\Delta_\sigma \subset \mathcal{L}^2_\sigma$ and is isometric. The inversion of equality (3) has the form

$$f(x) = \underset{n\to\infty}{\text{l.i.m.}} \int_{-n}^{n} F(\lambda)\overline{P(x,\lambda)}\, d\sigma(\lambda). \tag{4}$$

The simplest example of a family $P(x,\lambda)$ is given by the Fourier integral: $P(x,\lambda) = e^{ix\lambda}$ $(x \geq 0)$, $\sigma(\lambda) = \dfrac{1}{2\pi}\lambda$ $(-\infty < \lambda < \infty)$.

M. G. Kreĭn ([1]) showed that to every distribution function $\sigma(\lambda)$ satisfying relation (1) and some further rather general conditions there corresponds uniquely a family $P(x,\lambda)$ with precisely the properties just listed. The general method of constructing the function $P(x,\lambda)$ from the distribution function $\sigma(\lambda)$ is not very effective and, like the analogous method in the inverse Sturm–Liouville problem, does not make it possible to clarify the influence of empty intervals in the spectrum of the function $\sigma(\lambda)$, if any exist.

The present article is devoted to one simple but, it seems to me, important case of this kind. Before turning to it, let us note that, together with $P(x,\lambda)$, it is useful to consider the function $P_*(x,\lambda) = e^{ix\lambda}\overline{P(x,\lambda)}$.

One of M. G. Krein's results is that the functions $P(x,\lambda)$, $P_*(x,\lambda)$ are connected by the differential relations

$$\frac{dP(x,\lambda)}{dx} = i\lambda P(x,\lambda) + D(x)P_*(x,\lambda), \qquad \frac{dP_*(x,\lambda)}{dx} = \overline{D(x)}P(x,\lambda).$$

2. Denote by E the real axis of the λ-plane with the interval $(-1,1)$ removed from it, and take an absolutely continuous distribution function $\sigma(\lambda)$, whose derivative is equal to

$$\sigma'(\lambda) = \frac{1}{2\pi}\sqrt{\frac{\lambda+1}{\lambda-1}} \quad (\lambda \in E); \qquad \sigma'(\lambda) = 0 \quad (-1 < \lambda < 1).$$

For the construction of the family $P(x,\lambda)$ we shall apply the method of the papers ([2,3]), connected with certain considerations on the Riemann surface $\mathfrak{F}$, which is obtained from two copies $(\mathfrak{G}, \mathfrak{G}^*)$ of the λ-plane, cut along the intervals $(-\infty,-1)$, $(1,\infty)$, by gluing, under which these intervals become transition lines. The radicals $\sqrt{\lambda^2-1}$, $\sqrt{\dfrac{\lambda+1}{\lambda-1}}$ will be regarded as positive on the upper bank of the cut $(1,\infty)$ of the sheet $\mathfrak{G}$. With the aid of the method mentioned we find that in the case under consideration

$$P(x,\lambda) = \frac{1}{2}\left(1 + \sqrt{\frac{\lambda-1}{\lambda+1}}\right) e^{\frac{1}{2}ix(\lambda+\sqrt{\lambda^2-1})} + \frac{1}{2}\left(1 - \sqrt{\frac{\lambda-1}{\lambda+1}}\right) e^{\frac{1}{2}ix(\lambda-\sqrt{\lambda^2-1})},$$

whence it is not difficult to derive that

$$\frac{dP(x,\lambda)}{dx} = i\lambda P(x,\lambda) + \frac{1}{2i}P_*(x,\lambda), \qquad \frac{dP_*(x,\lambda)}{dx} = -\frac{1}{2i}P(x,\lambda).$$

Thus, in the case under consideration $D(x)$ is a constant.

We restrict ourselves to verifying Parseval' s equality (2). Let $f(x)$ $(x \geq 0)$ be an arbitrary continuous finite function, which we may assume to be equal to zero in some neighborhood of the point $x = 0$, say for $0 \leq x \leq \delta$.

Introduce the usual Fourier transforms

$$\varphi(z) = \int_0^\infty f(x)e^{ixz}\,dx, \qquad \psi(z) = \int_0^\infty \overline{f(x)}e^{-ixz}\,dx.$$

We can write the equality

$$F(\lambda) = \int_0^\infty f(x)P(x,\lambda)\,dx =$$

$$= \frac{1}{2}\left(1+\sqrt{\frac{\lambda-1}{\lambda+1}}\right)\varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right) + \frac{1}{2}\left(1-\sqrt{\frac{\lambda-1}{\lambda+1}}\right)\varphi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right),$$

and, on the other hand, on the basis of Plancherel' s theorem,

$$\int_0^\infty |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty |\varphi(t)|^2\,dt. \tag{5}$$

Our task now consists in expressing the right-hand side of this formula through $F(\lambda)$. To this end note that for $\lambda \in E$

$$|F(\lambda)|^2 = \frac{1}{2(\lambda+1)}\left[(\lambda+\sqrt{\lambda^2-1})\left|\varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\right|^2 +\right.$$

$$\left.+(\lambda-\sqrt{\lambda^2-1})\left|\varphi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\right|^2\right] + \frac{1}{2(\lambda+1)}\left[\varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\psi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right) +\right.$$

$$\left.+\varphi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\psi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\right].$$

We must compute, along the upper bank of E on the sheet $\mathfrak{G}$, the integral

$$\int_E |F(\lambda)|^2\, d\sigma(\lambda) = J_1 + J_2.$$

First of all let us show that $J_2 = 0$. Indeed,

$$J_2 = \frac{1}{2\pi}\operatorname{Re}\int_E \varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\psi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\frac{d\lambda}{\sqrt{\lambda^2-1}} =$$

$$= \frac{1}{2\pi}\operatorname{Re}\int_{-\infty}^{\infty} \varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\psi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\frac{d\lambda}{\sqrt{\lambda^2-1}}.$$

Applying the residue theorem, we find that

$$J_2 = -\frac{1}{2\pi}\operatorname{Re}\lim_{N\to\infty}\int_{K_N} \varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\psi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\frac{d\lambda}{\sqrt{\lambda^2-1}},$$

where K_N is the semicircle $\lambda = Ne^{i\theta}$ $(0 \le \theta \le \pi)$. On this semicircle

$$\varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right) = e^{-\delta N \sin\theta} O(1), \qquad \psi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right) = O(1),$$

and therefore $J_2 = 0$. We now turn to the integral $J_1 = J_1' + J_1''$, where

$$J_1' = \frac{1}{4\pi}\int_E \left|\varphi\left(\frac{\lambda+\sqrt{\lambda^2-1}}{2}\right)\right|^2 \left(1 + \frac{\lambda}{\sqrt{\lambda^2-1}}\right) d\lambda,$$

$$J_1'' = -\frac{1}{4\pi}\int_E \left|\varphi\left(\frac{\lambda-\sqrt{\lambda^2-1}}{2}\right)\right|^2 \left(1 - \frac{\lambda}{\sqrt{\lambda^2-1}}\right) d\lambda.$$

Obviously,

$$J_1' = \frac{1}{2\pi}\left(\int_{-\infty}^{-1/2} + \int_{1/2}^{\infty}\right)|\varphi(t)|^2\, dt, \qquad J_1'' = -\frac{1}{2\pi}\left(\int_0^{-1/2} + \int_{1/2}^{0}\right)|\varphi(t)|^2\, dt,$$

and, in order to prove Parseval’ s equality (2), it remains to recall (5).

If there is an empty interval in the spectrum, the operator U, introduced in No. 1, is unitary, i.e. $\Delta U = \mathcal{L}^2_\sigma$. Therefore, from the considerations of the present section there follows

Theorem 1. If $F(\lambda)$ is measurable and

$$\int_E |F(\lambda)|^2 \sqrt{\frac{\lambda+1}{\lambda-1}}\, d\lambda < \infty,$$

then

$$F(\lambda) = \int_0^\infty f(x) \left\{ \frac{1}{2}\left(1+\sqrt{\frac{\lambda-1}{\lambda+1}}\right) e^{\frac{1}{2}ix(\lambda+\sqrt{\lambda^2-1})} + \right.$$

$$\left. +\frac{1}{2}\left(1-\sqrt{\frac{\lambda-1}{\lambda+1}}\right) e^{\frac{1}{2}ix(\lambda-\sqrt{\lambda^2-1})} \right\} dx,$$

$$\frac{1}{2\pi}\int_E |F(\lambda)|^2 \sqrt{\frac{\lambda+1}{\lambda-1}}\, d\lambda = \int_0^\infty |f(x)|^2\, dx.$$

3. Let us generalize the proposition just proved, following the same path as in (4).

Lemma. Every polynomial $\Omega(\lambda)$, positive on the set E, can be represented, and in a unique way, in the form

$$\Omega(\lambda) = \omega(\lambda)\omega^*(\lambda),$$

where

$$\omega(\lambda) = \sum_{k=-n}^{n} a_k(\lambda+\sqrt{\lambda^2-1})^k, \qquad \omega^*(\lambda) = \sum_{k=-n}^{n} \overline{a_k}(\lambda-\sqrt{\lambda^2-1})^k$$

and $a_{-k} = \overline{a_k}$, while all zeros of $\omega(\lambda)$ lie on the sheet $\mathfrak{S}^*$, and all zeros of $\omega^*(\lambda)$ on the sheet $\mathfrak{S}$; $2n$ is the degree of the polynomial $\Omega(\lambda)$.

Consider the space $\mathcal{L}^2_\tau$, where $\tau(\lambda)$ is absolutely continuous and

$$\tau'(\lambda) = \frac{1}{\Omega(\lambda)}\sigma'(\lambda),$$

so that

$$\tau'(\lambda) = \frac{1}{2\pi}\frac{1}{\Omega(\lambda)}\sqrt{\frac{\lambda+1}{\lambda-1}} \quad (\lambda \in E); \qquad \tau'(\lambda) = 0 \quad (-1 < \lambda < 1),$$

where $\Omega(\lambda)$ is some polynomial of degree $2n$, positive on E.

Let us find its representation according to the lemma and construct the family of functions

$$Q(x,\lambda) = \frac{1}{2}\left(1 + \sqrt{\frac{\lambda-1}{\lambda+1}}\right)\omega(\lambda)e^{\frac{1}{2}ix(\lambda+\sqrt{\lambda^2-1})} +$$

$$+\frac{1}{2}\left(1 - \sqrt{\frac{\lambda-1}{\lambda+1}}\right)\omega^*(\lambda)e^{\frac{1}{2}ix(\lambda-\sqrt{\lambda^2-1})}.$$

The function $Q(x,\lambda)$, for every $x \geq 0$, is entire with respect to λ, and moreover of degree x. Let us consider polynomials $Q_k(\lambda)$ of degree $k = 0, 1, 2, \dots, n-1$, orthonormal in the space $\mathcal{L}^2_\tau$.

Theorem 2. *A function $F(\lambda)$ $(\lambda \in E)$ belongs to $\mathcal{L}^2_\tau$ if and only if it admits the representation*

$$F(\lambda) = \sum_{k=0}^{n-1} c_k Q_k(\lambda) + \int_0^\infty f(x)Q(x,\lambda)\,dx, \tag{6}$$

$$f(x) \in \mathcal{L}^2(0,\infty).$$

In this case

$$\int_E |F(\lambda)|^2\,d\tau(\lambda) = \sum_{k=0}^{n-1}|c_k|^2 + \int_0^\infty |f(x)|^2\,dx.$$

To prove the necessity of the representation (6) for $F(\lambda)$ to belong to the space $\mathcal{L}^2_\tau$, let us find, for the given function $F(\lambda) \in \mathcal{L}^2_\tau$, the coefficients

$$c_k = \int_E F(\lambda)Q_k(\lambda)\,d\tau(\lambda) \qquad (k = 0, 1, \dots, n-1)$$

and introduce the function

$$G(\lambda) = \frac{1}{\Omega(\lambda)}\left[F(\lambda) - \sum_{k=0}^{n-1} c_k Q_k(\lambda)\right], \tag{7}$$

which, evidently, belongs to the space $\mathcal{L}^2_\sigma$ together with its product by any polynomial of degree $\leq n$ in λ, and

$$\int_E G(\lambda)\lambda^k \, d\sigma(\lambda) = 0 \qquad (k = 0, 1, \dots, n-1). \tag{8}$$

Representing $G(\lambda)$ in the form

$$G(\lambda) = \int_0^\infty g(x) P(x, \lambda) \, dx,$$

let us consider the corresponding $\mathcal{L}^2_\tau$ function

$$\Omega(\lambda) G(\lambda) = \omega(\lambda)\omega^*(\lambda) \int_0^\infty g(x) P(x, \lambda) \, dx. \tag{9}$$

Taking into account the relations (8), as well as the structure of the functions $\omega(\lambda)$, $\omega^*(\lambda)$, we find, after integration by parts, that

$$\int_0^\infty g(x)\omega^*(\lambda) e^{\frac{1}{2}ix(\lambda+\sqrt{\lambda^2-1})} \, dx = \int_0^\infty f(x) e^{\frac{1}{2}ix(\lambda+\sqrt{\lambda^2-1})} \, dx,$$

$$\int_0^\infty g(x)\omega(\lambda) e^{\frac{1}{2}ix(\lambda-\sqrt{\lambda^2-1})} \, dx = \int_0^\infty f(x) e^{\frac{1}{2}ix(\lambda-\sqrt{\lambda^2-1})} \, dx,$$

where on the right-hand sides there is one and the same function $f(x) \in \mathcal{L}^2(0, \infty)$.

The equality (9) can now be written in the form

$$\Omega(\lambda) G(\lambda) = \int_0^\infty f(x) Q(x, \lambda) \, dx,$$

which, together with (7), proves the representation (6).

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Note: Figure translations are in progress. See original paper for figures.