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Abstract

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ON SOME BASIC INEQUALITIES IN THE THEORY OF ORTHOGONAL POLYNOMIALS

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1. Let the polynomials $\{\varphi_n(z)\}_0^\infty$ be orthonormal on the circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ with respect to the distribution $d\sigma(\theta)$; consider three inequalities

$$\sigma'(\theta_0) \ge m > 0; \tag{I}$$

$$K_n(z_0) = \sum_{i=0}^{n} |\varphi_i(z_0)|^2 \le C(n+1), \qquad z_0 = e^{i\theta_0}; \tag{II}$$

$$\{|\varphi_n(z_0)|\}_0^\alpha \le M. \tag{III}$$

Theorem 1. *If $\sigma'(\theta_0)$ exists and if*

$$\int_0^h |dt\{\sigma(\theta_0 + t) - \sigma(\theta_0 - t) - 2t\sigma'(\theta_0)\}| = o(h), \tag{1}$$

then (I) follows from (II).

The proof follows from the inequality $(n+1) \le K_n(z_0)\sigma_n(\theta_0)$, where $\sigma_n(\theta)$ is the Fejér sum for the Fourier-Stieltjes series $\mathfrak{S}(d\sigma)$; from condition (1) there follows the convergence $\lim_{n\to\infty} \sigma_n(\theta_0) = \sigma'(\theta_0)$.

Theorem 1 is also valid in the more general case when (II) holds only for a subsequence $\{K_{n_\nu}(z_0)\}$, and also in the case when there exists only the generalized derivative

$$\sigma^{(1)}(\theta_0) = \lim_{h\to 0} \frac{\sigma(\theta_0 + h) - \sigma(\theta_0 - h)}{2h}. \tag{2}$$

We see that (II) implies (I); it is also obvious that (III) implies (II), and consequently also (I),—but does (III) follow from (I) or from (II)? This question was posed in its time by V. A. Steklov (*1*); in our works (*2*, *3*) we obtained some sufficient conditions for the boundedness of the whole orthonormal system on the entire interval of orthogonality or on a part of it, but in doing so we had to impose on the function $\sigma(\theta)$ certain restrictions not only of a local but also of a global character, and the question remained open.

2. We shall use the relations

$$\sqrt{1-|a_n|^2}\,\varphi_{n+1}(z) = z\varphi_n(z) - \bar{a}_n\varphi_n^*(z),$$

$$\sqrt{1-|a_n|^2}\,\varphi_{n+1}^*(z) = \varphi_n^*(z) - a_n z\varphi_n(z),$$

$$\varphi_n^*(z) = z^n\overline{\varphi_n\left(\frac{1}{z}\right)} \qquad (n=0,1,...), \tag{3}$$

where the parameters $\{a_n\}_0^\infty$ are subject to the conditions $\{|a_n|\}_0^\infty < 1$; conversely, if the parameters are chosen completely arbitrarily and independently of one another

one another, subjecting them only to these conditions, then the polynomials $\{\varphi_n(z)\}_0^\infty$, constructed by the recurrence formula (3), are orthonormal on $|z|=1$ with respect to a certain distribution $d\sigma(\theta)$, which is completely determined by these parameters. From (3) there follows the formula

$$\sqrt{1-|a_n|^2} = \frac{2\Re\eta_n}{1+|\eta_n|^2} \leqslant \frac{2|\eta_n|}{1+|\eta_n|^2}, \qquad \eta_n = \frac{z_0\varphi_n(z_0)}{\varphi_{n+1}(z_0)}, \qquad z_0 = e^{i\theta_0}. \tag{4}$$

Theorem 2. Under condition (II) one has $\underline{\lim}_{n\to\infty}|\varphi_n(z_0)| < \infty$; if at the same time $\overline{\lim}_{n\to\infty}|\varphi_n(z_0)| = \infty$, then $\lim_{n\to\infty}|a_n| = 1$.

If we had $\lim_{n\to\infty}|\varphi_n(z_0)| = \infty$, then from this would follow the existence of the limits $\lim_{n\to\infty}|\varphi_n(z_0)| = \infty$ and $\lim_{n\to\infty}K_n(z_0)/(n+1) = \infty$, which contradicts condition (II).

Suppose that $\overline{\lim}_{n\to\infty}|\varphi_n(z_0)| = \infty$; we divide the set of all natural numbers into two sequences $\{n_i'\}$ and $\{n_k''\}$ so as to have

$$|\varphi_{n_i'}(z_0)| \leqslant A \quad (i=1,2,...); \qquad \lim_{k\to\infty}|\varphi_{n_k''}(z_0)| = \infty; \tag{5}$$

both sequences are, obviously, infinite.

We shall consider pairs of numbers n and $n+1$ belonging to two different sequences–such pairs will always be found for arbitrarily large values of n, since, if beginning with $n = n_0$ no such pairs were found, then one of the sequences would have no more than n_0 terms. Choose n so large that $|\eta_n| < \varepsilon$ or $|\eta_n| > 1/\varepsilon$, where $\varepsilon > 0$ is arbitrarily small; in both cases we obtain from (2) $\sqrt{1-|a_n|^2} < 2\varepsilon$, which is equivalent to the condition $\lim_{n\to\infty} |a_n| = 1$.

Remark 1. Formula (4) shows that, knowing $\varphi_n(z_0)$, we may choose $\varphi_{n+1}(z_0) \neq 0$ completely arbitrarily—this determines $|a_n|$, and by formula (3) also a_n; under condition (II) we shall indicate several examples of such a choice:

1) Put $\varphi_m(z_0) = C\sqrt{m}$ for $m = 2^k$ $(k = 0, 1, 2, ...)$ and $\varphi_m(z_0) = 1$ for all other values of m.

2) Let $\varphi_m(z_0) = C\sqrt[3]{m}$ for $m = k^3$ $(k = 1, 2, 3, ...)$ and $\varphi_m(z_0) = 1$ for all other values of m; it can be shown that in both cases we have

$$\lim_{n\to\infty} \left[\prod_{k=0}^{n-1}(1-|a_k|^2)\right]^{1/2} = 1,$$

whence it follows that the set of points of increase of the function $\sigma(\theta)$ is everywhere dense on the interval $[0, 2\pi]$.

3. **Theorem 3.** 1) From the validity of (II) almost everywhere on $[\alpha, \beta]$ there follows the validity of (I) almost everywhere inside $[\alpha, \beta]$.

2) From the validity of (I) almost everywhere on $[\alpha, \beta]$ there follows the validity of (II) inside $[\alpha, \beta]$.

The proof of assertion 1) follows from the convergence $\lim_{n\to\infty} \sigma_n(\theta) = \sigma'(\theta)$ almost everywhere on $[\alpha, \beta]$; assertion 2) is known.

Theorem 4. From the validity of (I) almost everywhere on $[\alpha, \beta]$ there follows the existence of an infinite subsequence of polynomials $\{\varphi_{n_\nu}(e^{i\theta})\}$, bounded at every point inside $[\alpha, \beta]$, uniformly bounded on some interior subinterval; under the additional condition $\overline{\lim}_{n\to\infty} |a_n| < 1$ the entire sequence $\{\varphi_n(e^{i\theta})\}_0^\infty$ has this property.

Indeed, from the proof of Theorem 2 it follows that, under the condition $\overline{\lim}_{\nu\to\infty}|a_\nu| < 1$, we have $\overline{\lim}_{\nu\to\infty}|\varphi_\nu(e^{i\theta})| < \infty$ at every point at which (11) is satisfied; the uniform boundedness follows from Szegő' s theorem.

Remark 2. The condition $\overline{\lim}_{n\to\infty}|a_n| < 1$ is satisfied, in particular, if $\lg \sigma'(\theta) \in L_1$, since this latter condition is equivalent to the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty;$$

but it is also satisfied in a number of other cases—for example, in the case considered by N. I. Akhiezer ([4]), since in this case there exists the limit $\lim_{n\to\infty} a_n = a,\ |a| < 1$.

4. From the theorems proved there follow analogous theorems for polynomials $\{p_n(x)\}_0^\infty$ orthonormal on the interval $[-1,+1]$ with respect to the distribution $d\psi(x)$; here the condition $\overline{\lim}_{n\to\infty}|a_n| < 1$ is equivalent to the condition $\underline{\lim}_{n\to\infty}\lambda_n > 0$, where λ_n is the coefficient in the recurrence formula

$$\sqrt{\lambda_{n+1}}\,p_n(x) = (x-a_n)p_{n-1}(x) - \sqrt{\lambda_n}\,p_{n-2}(x) \qquad (n=1,2,...); \quad (6)$$

the condition $\underline{\lim}_{n\to\infty}\lambda_n > 0$ is satisfied, in particular, if $\lg\psi'(x)/\sqrt{1-x^2} \in L_1$, and is also satisfied in a number of other cases, for example, in the case of Pollaczek polynomials, for which $\lg\psi'(x)/\sqrt{1-x^2} \notin L_1$, but $\lim_{n\to\infty}\lambda_n = 1/4$.

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Note: Figure translations are in progress. See original paper for figures.