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Abstract

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ON EXPANSION IN EIGENFUNCTIONS OF A NONSELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS OF SECOND ORDER

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The present work is devoted to the study of expansion in eigenfunctions of a nonselfadjoint system of differential equations in the space of vector-functions. The scalar case was first studied in the fundamental work of M. A. Naimark ([1]).

Consider a system of differential expressions of second order, which is written briefly in the form

$$l(y) = -y'' + P(x)y, \tag{1}$$

where $y(x) = (y_1(x), \dots, y_k(x))$ is a vector-function; $P(x)$ is a k-dimensional complex-valued matrix, summable on the interval $[0, \infty]$.

With this differential expression we construct an operator in the space of vector-functions $y(x) \in L_k^2(0, \infty)$.

$y(x) \in L_k^2(0, \infty)$, if

$$\int_0^\infty \sum_{i=1}^k |y_i(x)| \, dx < \infty.$$

Denote by D the set of those vector-functions $y(x) \in L_k^2(0, \infty)$ such that: 1) the derivative of the vector-function $y'(x)$ exists and is absolutely continuous on every finite interval $[0, b]$, $b > 0$; 2) $l(y) \in L_k^2(0, \infty)$. By D_Θ denote the set of vector-functions $y(x) \in D$ satisfying the condition

$$y'(0) - \Theta y(0) = 0, \tag{2}$$

where Θ is a fixed k-dimensional complex matrix.

Define the operator L_Θ as follows: its domain of definition is D_Θ, and for $y(x) \in D_\Theta$, $L_\Theta y = l(y)$.

Denote by $Y_1(x,s)$ and $Y_2(x,s)$ $(s^2=\lambda)$ linearly independent solutions of the matrix equation

$$-Y''+P(x)Y=\lambda Y. \tag{3}$$

These solutions are constructed so that they satisfy the following asymptotic formulas:

as $x\to\infty$,

$$Y_1(x,s)=e^{isx}[1+o(1)]$$

uniformly with respect to s, $|s|\geq r>0$, $\operatorname{Im} s\geq 0$;

$$Y_2(x,s)=e^{-isx}[1+o(1)]$$

uniformly with respect to s, $|s|\geq r>0$, $\operatorname{Im} s\leq 0$;

as $s\to\infty$,

$$Y_1(x,s)=e^{isx}\left[1+O\left(\frac{1}{s}\right)\right], \qquad Y_2(x,s)=e^{-isx}\left[1+O\left(\frac{1}{s}\right)\right]$$

uniformly with respect to x, $0\leq x<\infty$.

Analogous solutions $Z_1(x,s)$ and $Z_2(x,s)$ are constructed for the matrix equation

$$-Z''+ZP(x)=\lambda Z. \tag{4}$$

Denote by $\xi_1(s),\dots,\xi_k(s)$ the eigenvalues of the matrix

$$|A_1(s)-\xi A_2(s)|=0,$$

where

$$A_1(s)=Y_1'(0,s)-\Theta Y_1(0,s), \tag{5}$$

$$A_2(s)=Y_2'(0,s)-\Theta Y_2(0,s), \tag{6}$$

and by $\rho_1(x),\dots,\rho_k(s)$ the corresponding eigenvectors. Let $\xi_1'(s),\dots,\xi_k'(s)$ be the eigenvalues, and $\rho_1'(s),\dots,\rho_k'(s)$ the corresponding eigenvectors of the matrix

$$B_1(s)-\xi' B_2(s)=0,$$

where

$$B_1(s)=Z_1'(0,s)-Z_1(0,s)\Theta, \tag{7}$$

$$B_2(s)=Z_2'(0,s)-Z_2(0,s)\Theta. \tag{8}$$

If the matrix $P(x)$ is summable on the interval $[0, \infty]$, then the following holds:

Theorem 1. The spectrum of the operator L_Θ is continuous on the positive half-axis and discrete in the entire remaining complex λ-plane. The eigenvalues of the operator L_Θ form a bounded set, whose limit points may lie only on the positive half-axis $\lambda \geq 0$. For values of λ not belonging to the spectrum, the resolvent of the operator L_Θ is an integral operator with kernel $K(x, \xi, \lambda)$, satisfying the conditions:

$$\int_0^\infty |K(x, \xi, \lambda)|^2 \, d\xi < \infty, \qquad \int_0^\infty |K(x, \xi, \lambda)|^2 \, dx < \infty.$$

Theorem 1 follows from the asymptotic behavior of $Y_1(x, s)$ and $Y_2(x, s)$ for large s. In what follows we assume that

$$\int_0^\infty x^2 |P(x)| \, dx < \infty$$

and that:

1) the eigenvalues of the operator L_Θ are simple poles of its resolvent; 2) the matrices $A_1(s)$ and $A_2(s)$ are non-singular for $s \geq 0$. In this case the discrete part of the spectrum consists of a finite number of points, and the point $\lambda = 0$ is not a spectral point. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the eigenvalues, and $y_1(x), y_2(x), \ldots, y_r(x)$ the corresponding eigenvector-functions of the operator L_Θ.

Theorem 2. If conditions 1) and 2) are satisfied, then for any point λ not belonging to the spectrum of the operator L_Θ,

$$K(x, \xi, \lambda) = \sum_{j=1}^{r} \frac{y_j(x) z_j^*(\xi)}{(\lambda_j - \lambda) \displaystyle\int_0^\infty (y_j, z_j) \, dx} - \frac{1}{\pi} \int_0^\infty \sum_{j=1}^{k} \frac{[Y_1(x, s) - \xi_{jY} 2(x, s)] \rho_j \rho_j'^* [z_1(\xi, s) - \xi_{jz}' 2(\xi, s)]}{(s^2 - \lambda)[\xi_j(s) + \xi_j'(s)](\rho_j, \rho_j')} \, ds, \tag{9}$$

where the integral on the right converges absolutely and uniformly with respect to x, ξ in the domain $0 \leq x, \xi < \infty$.

Theorem 3. If conditions 1) and 2) are satisfied, then every vector-function $g(x) \in D_\Theta$ can be represented in the form

$$g(x) = \sum_{j=1}^{r} \frac{y_j(x) \displaystyle\int_0^\infty (g, z_j) \, dx}{\displaystyle\int_0^\infty (y_j, z_j) \, dx} - \frac{1}{\pi} \int_0^\infty \sum_{j=1}^{k} \frac{[Y_1(x, s) - \xi_j(s) Y_2(x, s)] \rho_j \rho_j'^* F_j(s)}{[\xi_j(s) + \xi_j'(s)](\rho_j, \rho_j')} \, ds. \tag{10}$$

where

$$F_j(s) = \int_0^\infty [z_1(\xi, s) - \xi_j'(s) z_2(\xi, s)] g(\xi)\, d\xi.$$

The integral on the right converges absolutely and uniformly with respect to x in the interval $0 \le x < \infty$.

There is an analogue of Parseval’ s equality.

Let $g(x) \in D_\Theta$ and $h(x) \in L_k^2(0, \infty)$; then

$$\int_0^\infty (g(x), h(x))\, dx = \sum_{j=1}^{r} \frac{\alpha_j \beta_j}{\int_0^\infty (y_j, z_j)\, dx} - \frac{1}{\pi} \int_0^\infty \sum_{j=1}^{k} \frac{(\rho_j \rho_j^* F_j(s), H_j(s))}{[\xi_j + \xi_j'](\rho_j, \rho_j')}\, ds, \tag{11}$$

where

$$\alpha_j = \int_0^\infty (y_j, h)\, dx, \qquad \beta_j = \int_0^\infty (g, z_j)\, dx, \qquad H_j(s) = \int_0^\infty [Y_1^* - \bar{\xi}_{jY} 2^*] h(x)\, dx.$$

Theorem 3 is easily obtained from Theorem 2, and the analogue of Parseval’ s equality from Theorem 3.

We outline the proof of Theorem 2. First consider the case when the matrix $P(x)$ satisfies the condition

$$\int_0^\infty e^{\varepsilon x} |P(x)|\, dx < \infty \tag{12}$$

for some $\varepsilon > 0$. Then we pass to the case when the matrix $P(x)$ satisfies the condition

$$\int_0^\infty x^2 |P(x)|\, dx < \infty,$$

approximating it by matrices satisfying condition (12).

Consider the auxiliary boundary-value problem on the interval $[0, b]$, $b > 0$,

$$l(y) = \lambda y, \qquad y'(0) - \Theta y(0) = 0, \qquad y(b) = 0. \tag{13}$$

Let $K_b(x, \xi, \lambda)$ be the resolvent kernel of this boundary-value problem; then, as $b \to \infty$,

$$K_b(x,\xi,\lambda) = K(x,\xi,\lambda) + o(1) \tag{14}$$

uniformly with respect to x, ξ in every finite square $0 \le x, \xi \le c$, $c > 0$. For sufficiently large λ, to each of the eigenvalues $\lambda_1, \dots, \lambda_r$ of the operator L_Θ there corresponds exactly one eigenvalue $\lambda_1(b), \dots, \lambda_r(b)$ of the boundary-value problem (13), so that $\lambda_j(b) \to \lambda_j$, $j = 1, 2, \dots, r$, as $b \to \infty$. All remaining eigenvalues of the boundary-value problem, as $b \to \infty$, have the following asymptotics:

$$\lambda = s^2, \qquad s_{nj}^{(j)} = \frac{n\pi}{b} + \frac{1}{2bi} + \ln \xi_j\left(\frac{n\pi}{b}\right) + \frac{1}{b} o(1) \tag{15}$$

uniformly with respect to s in the domain $|\operatorname{Im} s| \le \varepsilon_1$, $\operatorname{Re} s \ge 0$. $y(x, s_n^{(j)}) = [Y_1(x,s) - \xi_j(s) Y_2(x,s)]\rho_j(s)$ are eigenfunctions corresponding to the eigenvalues (15).

If $y_j(x,b)$, $j = 1, 2, \dots, r$, are the eigenfunctions of the boundary-value problem (13), then as $b \to \infty$

$$\frac{y_j(x,b) z_j^*(\xi,b)}{\displaystyle\int_0^b (y_j, z_j)\, dx} = \frac{y_j(x) z_j^*(\xi)}{\displaystyle\int_0^\infty (y_j, z_j)\, dx} + o(1), \qquad j = 1, 2, \dots, r, \tag{16}$$

uniformly with respect to x, $0 \le x \le c$, $c > 0$.

Moreover, as $b \to \infty$,

$$\frac{1}{b} \int_0^\infty (y(x, s_n^{(j)}),\, z(x, s_n^{(j)}))\, dx = -\,[\xi_j(s) + \xi_j'(s)](\rho_j, \rho_j') + o(1) \tag{17}$$

uniformly with respect to s in every rectangle $|\operatorname{Im} s| \le \varepsilon_1$, $0 \le \operatorname{Re} s \le \beta$, $\beta > 0$.

For the derivation of formula (9), the kernel $K_b(x,\xi,\lambda)$ is considered, for $b = m\sqrt{q}$, on the contour $C_{m,q}$, which is chosen in the same way as in paper (1) (m, q are natural numbers). On the contour $C_{m,q}$, $|K(x,\xi,\lambda)| \le c/\sqrt{|\lambda|}$, and therefore

$$\frac{1}{2\pi i} \int_{C_{m,q}} \frac{K_b(x,\xi,\lambda)}{(\lambda - \lambda_0)}\, d\lambda \to 0 \quad \text{as } m \to \infty$$

uniformly with respect to q. Applying the residue theorem to the last integral and passing to the limit as $m \to \infty$, $q \to \infty$, taking into account (14), (16), and (17), we obtain formula (9) for $\lambda = \lambda_0$. If the eigenvalues $\lambda_1, \dots, \lambda_r$ of the operator L_θ have multiplicities $m_1, \dots, m_r$, respectively, then

$$K(x,\xi,\lambda) = \sum_{i=1}^{r} \sum_{p=1}^{m_i} \frac{G_p^{(i)}(x,\xi)}{(\lambda-\lambda_i)^p} - \frac{1}{\pi} \int_0^\infty \sum_{j=1}^{k} \frac{[Y_1 - \xi_{jY} 2]\rho_j \rho_j'^* [z_1 - \xi_j' z_2]}{(s^2-\lambda)[\xi_j + \xi_j'](\rho_j, \rho_j')}\, ds, \quad (18)$$

where

$$G_p^{(i)}(x,\xi) = -\sum_{j=0}^{m_i - p} y_j^{(i)}(x)\, z_{m_i-p-j}^{(i)*}(\xi);$$

for all p and i, $G_p^{(i)}(x,\xi)$ satisfies the boundary condition (2), and

$$l(y_j^{(i)}) - \chi_j y_j^{(i)} = y_{j-1}^{(i)}, \qquad l^*(z_j^{(i)}) - \overline{\lambda}_j z_j^{(i)} = z_{j-1}^{(i)},$$

where

$$y_{-1}^{(i)}(x) \equiv 0, \qquad z_{-1}^{(i)}(\xi) \equiv 0, \qquad \int_0^\infty (y_j^{(i)}, z_r^{(k)})\, dx = \delta_{ik}\delta_{m_i-j-1,r}.$$

The functions $y_0^{(i)}(x), \dots, y_{m_i-1}^{(i)}(x)$ form a chain of eigenfunctions and associated functions corresponding to the eigenvalue λ_i. If $P(x)$ satisfies condition (12), then assumption 2) can be dropped; in that case the integration is carried out over a contour obtained in the usual way by deforming the positive half-axis in the complex λ-plane. Another expansion is considered in paper (4).

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Note: Figure translations are in progress. See original paper for figures.

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