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Abstract

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MATHEMATICS

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ON THE SOLVABILITY OF THE DIRICHLET AND NEUMANN PROBLEMS FOR A LINEAR ELLIPTIC OPERATOR WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 14 X 1960)

This paper studies the question of the solvability (in the classical sense) of the Dirichlet and Neumann problems for a general linear elliptic operator of the second order with discontinuous coefficients.

1°. Let there be given an open N-dimensional domain g with boundary Γ, and inside it an $(N-1)$-dimensional surface C, homeomorphic to a sphere, dividing g into subdomains g_1 (lying inside C) and g_2. Let, further, T be some open domain containing within itself the closed domain $(g+\Gamma)$. Consider in the domain $(g+\Gamma)$ the following Dirichlet problem:

$$L_1 u = \sum_{i,k=1}^{N} a_{ik}^{(1)}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{N} b_i^{(1)}(x)\frac{\partial u}{\partial x_i} - c^{(1)}(x)u = f^{(1)}(x) \quad \text{in } g_1,$$

$$L_2 u = \sum_{i,k=1}^{N} a_{ik}^{(2)}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{N} b_i^{(2)}(x)\frac{\partial u}{\partial x_i} - c^{(2)}(x)u = f^{(2)}(x) \quad \text{in } g_2, \tag{1}$$

$$u\big|_{\Gamma} = \varphi, \qquad [u]\big|_{C} = \psi, \qquad \left[\frac{\partial u}{\partial \nu}\right]\Bigg|_{C} = \chi,$$

where

$$[u]\big|_C = u\big|_{C-0} - u\big|_{C+0}, \qquad \left[\frac{\partial u}{\partial \nu}\right]\Bigg|_C = \frac{\partial u}{\partial \nu_1}\Bigg|_{C-0} - \frac{\partial u}{\partial \nu_2}\Bigg|_{C+0};$$

ν_1 is the conormal, exterior with respect to the domain g_1, for the operator L_1; ν_2 is the conormal, interior with respect to the domain g_2, for the operator L_2; and the symbols $C-0$ and $C+0$ mean that limiting values are taken respectively from the inner and the outer, with respect to g_1, side of the surface C.

Definition. By a **classical solution of the Dirichlet problem** (1) we shall mean a function $u(x)$ satisfying the following conditions: 1) $u(x)$ belongs to the class $C^{(0)}$ in each of the closed domains $(g_1 + C)$ and $(g_2 + C + \Gamma)$, to the class $C^{(1)}$ in each of the domains $(g_1 + C)$ and $(g_2 + C)$, and to the class $C^{(2)}$ in each of the open domains g_1 and g_2; 2) $u(x)$ satisfies, in the classical sense, all the conditions of problem (1).

Suppose the following conditions are fulfilled: 1) the surface C belongs to the Lyapunov class, and the surface Γ is regular*; 2) the functions $a_{ik}^{(m)}(x)$, $b_i^{(m)}(x)$, $c^{(m)}(x)$, $f^{(m)}(x)$ $(m = 1, 2)$ are defined and belong to the classes: $a_{ik}^{(1)} \in C^{(1,\mu)}$ in the domain $(g_1 + C)$, $a_{ik}^{(2)}(x) \in C^{(1,\mu)}$ in $(T - g_1)$, $b_i^{(1)}(x)$ and $c^{(1)}(x) \in C^{(0,\mu)}$ in $(g_1 + C)$, $b_i^{(2)}(x)$ and $c^{(2)}(x) \in C^{(0,\mu)}$ in $(T - g_1)$, $f^{(1)}(x) \in C^{(0,\mu)}$ in g_1 and $\in C^{(0)}$ in $(g_1 + C)$, $f^{(2)}(x) \in C^{(0,\mu)}$ in g_2 and $\in C^{(0)}$ in $(g_2 + C + \Gamma)$; 3) the coefficients $a_{ik}^{(m)}$ $(m = 1, 2)$ everywhere in their domains of definition satisfy the ellipticity conditions; 4) $c^{(m)}(x) \geq 0$ everywhere in their domains of definition;

* That is, in the domain bounded by the surface Γ, the Dirichlet problem for the Laplace equation is solvable for any continuous boundary function.

5) the function $\varphi(x)$ is defined and continuous on the surface Γ; 6) the functions $\psi(x)$ and $\chi(x)$ are defined on the surface C and belong there to the following classes: $\psi \in C^{(1,\mu)}$ and $\chi \in C^{(0,\mu)}$. We shall call the six conditions indicated **conditions B**.

Theorem 1. *If the first, third, and fourth of conditions B are fulfilled, then there can exist at most one classical solution of the Dirichlet problem (1).*

This theorem is proved in exactly the same way as the corresponding theorem of [1].

Theorem 2. *If conditions B are fulfilled, then there exists a (and, moreover, unique) classical solution of the Dirichlet problem (1), and this solution belongs to the class $C^{(1,\delta)}$ in each of the domains $(g_1 + C)$ and $(g_2 + C)$.*

The question of solvability of the Dirichlet problem somewhat less general than (1) was studied in the works of O. A. Oleinik [2,3], but in those works solvability of the Dirichlet problem is established under the assumption that the coefficients of the equations and the boundaries of the domains satisfy extremely stringent (in comparison with conditions B) smoothness requirements, increasing without bound as the number N of dimensions increases.

Let us note that the solvability conditions for the Dirichlet problem (1), established in Theorem 2, in the special case when there are no discontinuities in the coefficients and when $\psi \equiv \chi \equiv 0$, coincide with the classical conditions of Giraud (see [4], p. 77). We emphasize that Giraud' s conditions are the sharpest of the hitherto known solvability conditions for the Dirichlet problem with smooth coefficients.

Remark 1. Theorems 1 and 2 extend to the case when inside the surface Γ there lie n closed surfaces of discontinuity of the coefficients $C_1, C_2, \ldots, C_n$ belonging

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to the Lyapunov class, and on each of these surfaces its own boundary conditions are prescribed:

$$[u]\big|_{C_i} = \psi_i, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_{C_i} = \chi_i \quad (i = 1, 2, \dots, n).$$

Here some of the surfaces C_i may lie inside others.

Remark 2. Results analogous to those formulated above have also been obtained by us for the Neumann problem with discontinuous coefficients, i.e. for a problem of the form (1), in which the condition $u|_\Gamma = 0$ is replaced by $(\partial u/\partial \nu_2 + hu)|_\Gamma = \varphi$, where $h(x) \geq 0$, $\varphi \in C^{(1,\mu)}$. If to the definition of a classical solution one adds $u(x) \in C^{(1)}$ in the closed domain $(g_2 + C + \Gamma)$, and to conditions B one makes the additions: 1) Γ is a surface of Lyapunov type; 2) $h(x) \in C^{(0)}$ on Γ, then for the Neumann problem theorems fully analogous* to Theorems 1 and 2 will be valid.

2°. For the case when $\varphi \equiv \psi \equiv \chi \equiv 0$ and the operators L_1 and L_2 are self-adjoint, we have studied the relation between the classical solution of the Dirichlet problem (1) and the generalized** solution of this problem.

With the aid of the results of [5] it is established elementarily that the classical solution of the indicated problem is at the same time a generalized one, i.e. it is established that the classical solution possesses first derivatives square-integrable over the closed domain $(g + \Gamma)$.

3°. Let us outline the scheme of the proof of Theorem 2. Relying on one result of Giraud [6], it is easy to reduce problem (1) to the following simpler problem:

$$\begin{gathered} L_1 v = 0 \quad \text{in } g_1; \qquad L_2 v = 0 \quad \text{in } g_2, \\ v\big|_\Gamma = 0, \qquad [v]\big|_C = 0, \qquad \left[\frac{\partial v}{\partial \nu}\right]\bigg|_C = \theta, \end{gathered} \tag{2}$$

where θ is a given function on the surface C, $\theta \in C^{(0,\mu)}$.

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* Of course, when $h(x) \equiv 0$ the uniqueness of the Neumann problem will hold only up to a constant term.

** For the definition of a generalized solution see [5].

Let y be any fixed point in C. By methods of potential theory it is easy to prove the existence of a classical solution of the following Dirichlet problem

$$\begin{gathered} L_1 w(x, y) = f(x, y) \quad \text{in } g_1, \\ w(x, y)\big|_{x \in C} = \varphi(x, y), \quad x \neq y, \end{gathered} \tag{3}$$

in which * $\varphi(x,y) \in N^{(\alpha)}$ $(\alpha > 1)$ for $x \in (g+\Gamma),\ y \in C;\ f(x,y) \in N^{(\beta)}$ $(\beta > 0)$ for $x \in (g_1 + C),\ y \in C$, and, in addition, $f(x,y)$ satisfies the Hölder condition in g_1 with respect to x.

Let us now consider, for $y \in C,\ x \in (g+\Gamma)$, the kernel $K_0(x,y)$ of the form:

$$K_0(x,y) = \begin{cases} K_2(x,y), & \text{for } x \in (g_2 + C + \Gamma),\ y \in C,\ x \neq y, \\ \widehat{K}_1(x,y), & \text{for } x \in (g_1 + C),\ y \in C,\ x \neq y, \end{cases} \tag{4}$$

where $K_2(x,y)$ is the Green function of the Dirichlet problem for the operator L_2, extended to the whole domain $(g+\Gamma)$, and $\widehat{K}_1(x,y)$ is the solution of problem (3) for $f(x,y) \equiv 0,\ \varphi(x,y) = K_2(x,y)$. We have succeeded in establishing a number of fine properties of the functions $\widehat{K}_1(x,y)$. In particular, we have proved that: 1) $\widehat{K}_1(x,y) \in N^{(2)}$ for $x \in (g_1 + C),\ y \in C$; 2) $\widehat{K}_1(x,y) = K_1(x,y)A(x,y) + B(x,y)$, where $K_1(x,y)$ is the principal fundamental solution for the operator L_1, extended to the whole space, and $A(x,y)$ is the solution of problem (3) for $f(x,y) \equiv 0,\ \varphi(x,y) = K_2(x,y)/K_1(x,y)$, and it is proved that $A(x,y)$, as a function of x, uniformly with respect to y, belongs to the class $C^{(0,\delta)}$ for $x \in (g_1 + C),\ y \in C;\ \partial A/\partial x_i \in N^{(N-1-\delta,\gamma)}$ for $x \in (g_1 + C),\ y \in C;\ \partial B/\partial x_i \in N^{(1+\delta,\gamma)}$ for $x \in (g_1 + C),\ y \in C$ (δ is some positive number, $\gamma < \delta$).

From the properties proved it follows elementarily that the function

$$v(x) = \int_C K_0(x,y)\mu(y)\, ds_y \tag{5}$$

for any bounded function $\mu(y)$ satisfies all the conditions of problem (2), except the last one. The requirement that this last condition be satisfied, as is not hard to verify, leads to the following integral equation for $\mu(y)$:

$$\mu(x) = \int_C F(x,y)\mu(y)\, ds_y + \theta_1(x). \tag{6}$$

The kernel $F(x,y)$ of this integral equation, by virtue of the above-noted properties of the function $\widehat{K}_1(x,y)$, belongs to the class $N^{(1+\delta,\gamma)}$. But then the Fredholm alternative is valid for equation (6). Moreover, every solution of this equation from the class $C^{(0)}$ also belongs to the class $C^{(0,\delta)}$.

By virtue of Theorem 1, the function $\mu_0(x)$, which is a solution of the homogeneous equation corresponding to equation (6), is equal to zero.

Thus it has been proved that equation (6), for any function $\theta_1 \in C^{(0,\delta)}$, has a unique solution $\mu(x) \in C^{(0,\delta)}$. After this it is obvious that the function $v(x)$, defined by formula (5), is a classical solution of problem (2).

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* Definitions of all classes used in this article may be found in (4).

Note: Figure translations are in progress. See original paper for figures.

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