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Abstract

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SOME ESTIMATES FOR SOLUTIONS OF SECOND-ORDER ELLIPTIC EQUATIONS

(Presented by Academician V. I. Smirnov, November 24, 1960)

In a finite domain Ω of the n-dimensional Euclidean space R_n with boundary $\Gamma\Omega$, consider the elliptic operator

$$\mathfrak{M} = \nabla[A(x)\nabla] + b(x)\nabla + c(x),$$

where $A(x)$ is a matrix whose first eigenvalue is not less than $\chi > 0$. We shall assume, for simplicity, that the coefficients $A(x)$, $b(x)$, $c(x)$ and the functions from the domain of definition of the operator $\mathfrak{M}$ are sufficiently smooth in $\overline{\Omega}$. However, the constants in the estimates formulated below do not depend on the moduli of continuity of the coefficients. The condition of sufficient smoothness is not essential and can be weakened.

In the paper (1), G. Stampacchia, under a condition ensuring uniqueness, obtained for a generalized solution in $\overset{\circ}{W}{}_2^{(1)}(\Omega)$ of the equation $\mathfrak{M}u = f$ the estimate*

$$\|u\|_{L_s(\Omega)} \le K\|f\|_{L_{r'}(\Omega)},$$

where $1 < r \le \dfrac{2n}{n+2}$, $\dfrac{1}{s} > \dfrac{1}{r} - \dfrac{2}{n}$, and the constant K does not depend on the moduli of continuity of the coefficients of the equation. He conjectured that the indicated estimate is also valid when $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{2}{n}$. The following theorem shows that Stampacchia’ s conjecture is valid for every $r \in (1, n/2)$.

Theorem 1. *The solution of the problem*

$$\mathfrak{M}u = f, \qquad u|_{\Gamma\Omega} = 0 \tag{1}$$

satisfies the inequality

$$\|u\|_{L_s(\Omega)} \le K\left[\|f\|_{L_r(\Omega)} + \|u\|_{L(\Omega)}\right], \tag{2}$$

where: 1) $\frac{1}{s} \geq \frac{1}{r} - \frac{2}{n}$ for $\frac{n}{2} > r$; 2) $s > 0$ is an arbitrary number for $r = \frac{n}{2}$, and

$$K = K\left(n, r, s, p_1, p_2, \chi, \|b(x)\|_{L_{p_1}(\Omega)}, \|c^+(x)\|_{L_{p_2}(\Omega)}, \operatorname{mes}_n \Omega\right) \quad \left(p_1 > n,\ p_2 > \frac{n}{2}\right)^{**}.$$

If the function $f(x)$ is summable to a power r smaller than $\frac{2n}{n+2}$, then in general it is impossible to estimate the solution of problem (1) in the space $\overset{\circ}{W}{}_2^1(\Omega)$ in terms of the norm of f in $L_{\frac{2n}{n+2}}(\Omega)$. In the following theorem there is obtained estimate of the gradient of the solution of problem (1) in the norm $L_q(\Omega)$, where $\frac{n}{n-1} < q < 2$.

Theorem 2. *The solution of problem (1) satisfies the inequality*

$$\|\nabla u\|_{L_s(\Omega)} \leq K\left[\|f\|_{L_r(\Omega)} + \|u\|_{L(\Omega)}\right], \tag{3}$$

where $\frac{1}{s} \geq \frac{1}{r} - \frac{1}{n}$, $\quad 1 < r < \frac{2n}{n+2}$, *and* K *depends on the same constants as in theorem 1.*

Let the boundary of the domain Ω be a smooth surface. Then analogous estimates hold for the problem

$$\mathfrak{M}u = f, \qquad \mathbf{n} \cdot A \cdot \nabla u + \beta u\big|_{\Gamma\Omega} = \varphi, \tag{4}$$

where β, φ are functions prescribed on $\Gamma\Omega$, and $\mathbf{n}$ is the outward normal to $\Gamma\Omega$.

Corresponding to inequality (2) is the estimate

$$\|u\|_{L_s(\Omega)} \leq K\left[\|f\|_{L_r(\Omega)} + \|\varphi\|_{L_t(\Gamma\Omega)} + \|u\|_{L(\Omega)}\right], \tag{5}$$

where: 1) $\frac{1}{s} \geq \frac{1}{r} - \frac{2}{n}$ for $\frac{n}{2} > r > 1$; 2) $s > 0$ is an arbitrary number for $r = \frac{n}{2}$; $t \geq \frac{r(n-1)}{n-r}$, and $K = K(n, r, s, t, p_1, p_2, p_3, \varkappa, \|\mathbf{b}(x)\|_{L_{p_1}(\Omega)}, \|c^+(x)\|_{L_{p_2}(\Omega)}, \|\beta^-(x)\|_{L_{p_3}(\Gamma\Omega)}, \Omega)$ $(p_1 > n,\ p_2 > \frac{n}{2},\ p_3 > n-1)$.

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* Here and below, by K various constants are denoted.

** $f^+(x)$ and $f^-(x)$ denote, respectively, the positive and negative parts of the function $f(x)$.

For the solution of problem (4) one can also prove the following inequality, analogous to inequality (3):

$$\|\nabla u\|_{L_s(\Omega)} \le K\left[\|f\|_{L_r(\Omega)} + \|\varphi\|_{L_t(\Gamma\Omega)} + \|u\|_{L(\Omega)}\right], \tag{6}$$

where $\frac{1}{s} \ge \frac{1}{r} - \frac{1}{n}$, $1 < r < \frac{2n}{n+2}$, $t > \frac{r(n-1)}{n-r}$, and $K = K(n, r, s, t, p_1, p_2, p_3, \varkappa, \|\mathbf{b}(x)\|_{L_{p_1}(\Omega)}, \|c^+(x)\|_{L_{p_2}(\Omega)}, \|\beta^-(x)\|_{L_{p_3}(\Gamma\Omega)}, \Omega)$ $(p_1 > n,\ p_2 > \frac{n}{2},\ p_3 > n-1)$.

The requirement of smoothness of the boundary of the domain Ω can be weakened. Estimates (5), (6) remain valid if it is assumed that $\Gamma\Omega$ is a piecewise smooth surface and that there exists a positive constant $\mathfrak{A}$, depending only on the domain Ω, such that for any closed set $E \subset \overline{\Omega}$ $(\operatorname{mes}_n E \le \frac{1}{2}\operatorname{mes}_n \Omega)$ with boundary ΓE the inequality

$$\operatorname{mes}_{n-1}(\Gamma\Omega \cap E) \le \mathfrak{A}\ \operatorname{mes}_{n-1}(\Omega \cap \Gamma E), \tag{7}$$

holds, where mes_{n-1} denotes $(n-1)$-dimensional Hausdorff measure.

It follows from the isoperimetric inequality that a domain satisfying condition (7) belongs to the class $J^{(n)}_{\frac{n-1}{n}}$, defined in note (^{3}).

Without condition (7), estimates (5), (6) are in general false. However, one can consider certain classes of domains and obtain estimates of solutions of problem (4) for these domains analogous to estimates (5), (6). We shall consider domains Ω satisfying the following condition. There exists a constant $\mathfrak{B}$, depending only on the domain Ω, such that for any closed set $E \subset \overline{\Omega}$ $(\operatorname{mes}_n E \le \frac{1}{2}\operatorname{mes}_n \overline{\Omega})$ with boundary ΓE, the inequality

$$\max\left\{\operatorname{mes}_n^{\alpha} E,\ \operatorname{mes}_{n-1}^{\beta}(\Gamma\Omega \cap E)\right\} \le \mathfrak{B}\ \operatorname{mes}_{n-1}(\Omega \cap \Gamma E), \tag{8}$$

holds, where $1 > \alpha > \frac{n-1}{n}$, $\beta > 1$, $(2\alpha - 1)\beta < \alpha$.

Example. For the n-dimensional cone $x_1^2 + \cdots + x_{n-1}^2 < x_n^{2\varkappa}$ $(1 < \varkappa < 2)$, condition (8) is fulfilled

$$\left(\alpha = \frac{\varkappa(n-1)}{\varkappa(n-1)+1},\ \beta = \frac{\varkappa(n-1)}{\varkappa(n-2)+1}\right).$$

We still assume that the boundary of the domain Ω is a piecewise smooth surface.

Theorem 3. *If the domain Ω satisfies condition (8), then the solution of problem (4) satisfies inequality (5), where $r > 1$, $t\beta[1 - r(1-\alpha)] > r\alpha$,*

$$\frac{1}{s} \ge \frac{1}{r} - 2(1-\alpha) \quad \text{for} \quad 2r(1-\alpha) < 1.$$

If $2r(1-\alpha)=1,\ r>1$, *then in inequality (5)* s *is an arbitrary positive number. Moreover,*

$$p_1(1-\alpha)>1,\quad 2p_2(1-\alpha)>1,\quad p_3[\alpha-\beta(2\alpha-1)]>\alpha.$$

Estimate (6) also remains valid if the exponents s,r,t,p_i *are changed as follows:* $r>1,\ s\le 2,\ t\beta[1-r(1-\alpha)]>r\alpha$,

$$\frac{1}{s}\ge\frac{1}{r}+\alpha-1,\quad p_1(1-\alpha)>1,\quad 2p_2(1-\alpha)>1,\quad p_3[\alpha-\beta(2\alpha-1)]>\alpha.$$

In the following theorem an estimate is formulated for the maximum of the modulus of the solution of problem (4) for domains of the class indicated above.

Theorem 4. *For the solution of problem (4) the estimate* *

$$|u|\le K\left[\|f\|_{L_r(\Omega)}+\|\varphi\|_{L_t(\Gamma\Omega)}+\|u\|_{L(\Omega)}\right], \tag{9}$$

is valid, where $2r(1-\alpha)>1,\ t[\alpha-\beta(2\alpha-1)]>\alpha$, *and*

$$K=K(r,t,\alpha,\beta,p_i,\varkappa,\|b(x)\|_{L_{p_1}(\Omega)},\|c^+(x)\|_{L_{p_2}(\Omega)},\|\beta^-(x)\|_{L_{p_3}(\Omega)},\mathfrak{B},\mathrm{mes}_n\,\Omega,\mathrm{mes}_{n-1}\,\Gamma\Omega)$$

$$(p_1(1-\alpha)>1,\ 2p_2(1-\alpha)>1,\ p_3[\alpha-\beta(2\alpha-1)]>\alpha).$$

If condition (7) is fulfilled, then in estimate (9) $t>n-1,\ 2r>n,\ p_1>n,\ 2p_2>n,\ p_3>n-1$.

We note that close results can also be obtained for unbounded domains.

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* Regarding estimates of the maximum of the modulus of the solution of equation (1), see articles ([1,2]).