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Abstract

Full Text

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ON THE THEORY OF HYPOELLIPTIC AND PARTIALLY HYPOELLIPTIC OPERATORS

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In the present note we consider the question of infinite differentiability, in all or in part of the variables, of solutions in the whole space of equations with constant coefficients. All the results of this note are obtained from the main theorem of ([7]), which describes the general form of solutions of a homogeneous equation, by applying the apparatus of spaces of type S and type $\mathcal{E}$ (see ([4],[6])).

The formulations of the results will be given in terms of spaces of type S, of type $\mathcal{E}$, and of their conjugates, since the class of these spaces is the most natural for a number of problems, in particular for ours. We shall now give brief definitions of these spaces.

The spaces S_α^β, introduced and studied in ([4]), are spaces of functions belonging to the Gevrey class of order β and decreasing at infinity, together with all their derivatives, no more slowly than an exponential of order $1/\alpha$, more precisely:

$$S_\alpha^\beta = \left\{ \varphi : \sup \left| \frac{D^q \varphi(x)}{B^q q^{q\beta}} \right| \leq C \exp\left(-A|x|^{1/\alpha}\right),\ \alpha > 0,\ \exists A,\ \exists B \right\}.$$

The spaces $\mathcal{E}_\alpha^\beta$, introduced in ([6]), are spaces of functions belonging to the Gevrey class of order β and growing, together with all their derivatives, more slowly than an exponential of order $1/\alpha$, namely:

$$\mathcal{E}_\alpha^{\beta(-)} = \left\{ \varphi : \sup_q \left| \frac{D^q \varphi(x)}{B^q q^{q\beta}} \right| \leq C \exp\left(A|x|^{1/\alpha}\right)\ \ \forall A\ \exists B\ (\forall B) \right\}.$$

The spaces S_∞^β, $\mathcal{E}_\infty^\beta$, S_α^∞, $\mathcal{E}_\alpha^\infty$, S_∞^∞, $\mathcal{E}_\infty^\infty$ are constructed in an analogous way; here the upper index ∞ means that the requirement of belonging to a Gevrey class is replaced by the requirement of infinite differentiability; the lower index ∞ means that, for spaces of type S, the requirement of exponential decrease is replaced by the requirement of faster-than-polynomial decrease, and, for spaces of type $\mathcal{E}$, growth not exceeding exponential is replaced by growth not exceeding polynomial.

We proceed to the formulation of the results. The following theorem complements the classical result of L. Hörmander, who singled out and gave an algebraic characterization of hypoelliptic operators, i.e. such operators for which all

SovietRxiv

generalized solutions of the corresponding homogeneous equations are infinitely differentiable. It turns out that if one imposes a restriction on the growth of solutions of a hypoelliptic equation, then they acquire additional smoothness; namely, the following holds:

Theorem 1. Let $p(D)$ be a hypoelliptic operator of index (order) γ (see ([6],[8])); then every solution of the equation

$$p(D)u = w, \qquad D = \left(i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}\right), \tag{1}$$

belonging to the space $S_\alpha^{\beta'}$, $1 = \alpha + \gamma\beta$, $0 < \alpha < 1$, is a function belonging to the Gevrey class of order $\dfrac{1-\alpha}{\gamma}$, more precisely, it belongs to $\mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}$, if the right-hand side w also belongs to this space.

Thus, if we consider only generalized, even of infinite order, solutions of (1), growing no faster than an exponential of order $\dfrac{1}{\alpha}$, then we obtain the Gevrey class of order $\dfrac{1-\alpha}{\gamma}$, narrower than in Hörmander in ([9]), where this order is equal to $\dfrac{1}{\gamma}$.

Conversely, if for some α and β, $1 < \alpha + \gamma\beta \leq \infty$, $0 < \alpha < 1$, all solutions of (1) with $w = 0$, belonging to $\mathcal{E}_\alpha^\beta$, also belong to $\mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}$, then the operator $p(D)$ is hypoelliptic of exponent not less than γ.

A special case of the first assertion of the theorem was also obtained by V. V. Grushin by methods different from ours.

The problem of hypoellipticity can be formulated in a somewhat more general way, namely: to describe all operators $p(D)$ such that for each of them there exist two different spaces Φ and $\Psi \subset \Phi$, belonging to the class of spaces of type S, type $\mathcal{E}$, and their duals, such that from the fact that $u \in \Phi$ is a solution of (1) with $w = 0$ it follows that $u \in \Psi$.

In addition to hypoelliptic operators, operators for which the corresponding polynomial $p(S)$ has a bounded set of real zeros also possess such a property. We shall call such operators **quasihypoelliptic**. For quasihypoelliptic operators, as is easy to show, every solution of (1) with $w = 0$, growing no faster than an exponential of order less than one, is an entire analytic function of order one; more precisely, every solution belonging to $S_\alpha^{\beta'}$, $1 < \alpha \leq \infty$, $0 \leq \beta \leq \infty$, also belongs to $\mathcal{E}_\alpha^0$. (For the case of the space S_∞^∞ see ([4]), Ch. III, § 2, item 4, p. 164.)

If, however, on the real space $p(s)$ vanishes only at the origin, then a stronger assertion is true, namely, every solution belonging to $S_\alpha^{\beta'}$, $1 < \alpha \leq \infty$, $0 \leq \beta \leq$

SovietRxiv

∞, also belongs to $\mathcal{E}_\alpha^{0-}$ and, consequently, is an entire function of order $\leq \dfrac{1}{\alpha}$ (see ([10])).

It turns out that **hypoelliptic and quasihypoelliptic operators exhaust all solutions of the problem formulated above**.

We now pass to the question of hypoellipticity with respect to some of the variables. We shall take as our basis the definition of operators hypoelliptic with respect to some of the variables given by Gårding and Malgrange in ([1]), and introduce constants characterizing such operators. Let y, z be some partition of the variables x into two groups. We shall call an operator $p(D)$ **hypoelliptic** in the variables y if, for every solution u of equation (1), we can define sections by the planes $y = c$ as certain generalized functions of z, and u, as a function of its sections, is infinitely differentiable. More precisely, let $\Phi = \Phi(x)$ be some space of generalized functions (in x); let $\Psi \supset \Phi$ and X be some spaces of infinitely differentiable functions, and let Ψ and X be reflexive. We shall call an operator $p(D)$ **hypoelliptic** in y if every solution (1) belonging to Φ also belongs to $\Psi(z)\widehat{\otimes}X(y)$ ($\widehat{\otimes}$ denotes the sign of the inductive tensor product, see ([2])); in other words, if it extends by continuity to a linear functional on $\Psi^*(z) \times X^*(y)$, which means precisely that the sections u by the planes $y = c$ are defined as functionals belonging to $\Psi(z)$, and u, as a function of its sections, belongs to $X(y)$.

Let ξ and η be groups of variables dual to the groups y and z.

Theorem 2a. *If the polynomial $p(s)$ does not vanish in some region of the form*

$$|\operatorname{Im} s|^{\frac{1}{\gamma}} + |\eta|^{\frac{1}{\mu}} \leq C(\xi) - B \tag{2}$$

with $0 < \gamma \leq \mu \leq 1$, then every solution of (1) belonging to $S_\alpha^{\beta^*}$, $1 = \alpha + \gamma\beta$, $0 < \alpha < 1$, also belongs to

$$S_\alpha^{\mu\beta^*}(z)\,\widehat{\otimes}\,\mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}(y).$$

Conversely, if for some α and β, $1 < \alpha + \gamma\beta \leq \infty$, $0 < \alpha < 1$, $0 < \gamma \leq \mu \leq 1$, every solution of (1) belonging to $\mathcal{E}_\alpha^\beta$ also belongs to

$$S_\alpha^{\frac{\mu(1-\alpha)^*}{\gamma}}(z)\,\widehat{\otimes}\,\mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}(y)$$

$$\left(\frac{1-\alpha}{\gamma} < \beta\right),$$

then the polynomial $p(s)$ does not vanish in some region of the form (2).

Theorem 2b. *If the polynomial $p(s)$ does not vanish in some region of the form (2) with $0 < \gamma \leq 1 \leq \mu \leq \infty$, then every solution of (1) belonging to $S_\alpha^{\beta^*}$, $\mu = \alpha\mu + \beta\gamma$, $0 < \alpha < 1$, also belongs to*

$$S_\alpha^{\beta^*}(z)\,\widehat{\otimes}\,\mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}(y).$$

SovietRxiv

Conversely, if for some α and β, $\mu < \alpha\mu + \beta\gamma$, $0 < \alpha < 1$, $0 < \gamma \leq 1 \leq \mu \leq \infty$, every solution of (1) belonging to $\mathcal{E}_\alpha^{\frac{\beta}{\mu}}$ also belongs to

$$S_\alpha^{\frac{\mu(1-\alpha)^*}{\gamma}}(z) \widehat{\otimes} \mathcal{E}_\alpha^{\frac{1-\alpha}{\gamma}}$$

$$\left(\frac{1-\alpha}{\gamma} < \frac{\beta}{\mu}\right),$$

then the polynomial $p(s)$ does not vanish in some region of the form (2).

An operator $p(D)$ for which the corresponding polynomial $p(s)$ does not vanish in a region of the form (2) will be called **hypoelliptic with respect to y of order γ and width μ.**

Corollary 1. *In order that the operator $p(D)$ be hypoelliptic with respect to y of order 1 and width 1 (width greater than 1), it is necessary and sufficient that the principal part of the polynomial $p(s)$ not vanish on the plane $\eta = 0$, except at the origin (vanish only on the plane $\xi = 0$).*

It is possible to construct exact formulas for computing γ and μ, analogous to formula (6) for computing the order of a hypoelliptic operator.

The union of operators each of which is hypoelliptic with respect to y, in the sense of our definition, with some order and width, is the class of operators hypoelliptic in the sense of Hörmander and Malgrange [1]. The union, over all γ, of operators hypoelliptic with respect to y and of width ∞ is the class of operators elliptic with respect to y in the sense of Ehrenpreis [3] and hypoelliptic with respect to Gorin [5]. Finally, "entirely elliptic" operators in the sense of Ehrenpreis [3] correspond to $\gamma = 1$ and $\mu = \infty$.

A further generalization of the notion of operators hypoelliptic in part of the variables can be obtained by assigning independent roles to all four quantities

$$|\operatorname{Im}\xi|,\ |\operatorname{Im}\eta|,\ |\operatorname{Re}\xi|,\ |\operatorname{Re}\eta|$$

in (2). Thus, for example, if one requires that the polynomial $p(s)$ have no roots in the region (2) with $\mu = \infty$ and $\operatorname{Im}\eta = 0$, then one obtains the class of operators hypoelliptic with respect to y in the sense of [11].

It seems natural to the author to call **hypoelliptic with respect to y of order γ** the operators satisfying Theorem 2 with $\mu = 1$.

Corollary 2. *If the operator $p(D)$ remains hypoelliptic with respect to y of order γ after every real rotation, then it is hypoelliptic of order γ (in the sense of [6,8]).*

If an operator that is hypoelliptic with respect to y after every rotation with some γ and width $\mu < 1$ is considered, then it is not necessarily hypoelliptic. Indeed, for example, the operator

$$i\partial/\partial x_1 + \partial^2/\partial x_2^2,$$

SovietRxiv

while not hypoelliptic, is hypoelliptic with respect to any of the variables after every real rotation with $\gamma = 1/2$ and $\mu = 1/2$.

Let us note one more consequence of Theorem 2: if y is a variable with respect to which the operator $p(D)$ is hypoelliptic, then one may consider the Cauchy problem with y as the time axis for solutions of (1) that are generalized functions in all variables.

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