MATHEMATICS

1961

SovietRxiv

Abstract

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MATHEMATICS

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A CONDITION FOR THE EXPANDABILITY OF FUNCTIONS IN A QUASI-POWER SERIES IN THE BASIC INTERVAL

(Presented by Academician M. V. Keldysh on 10 VII 1950)

In the paper ([1]) a condition was established for the expandability of functions of certain quasi-analytic classes in the quasi-power series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n(u, t), \tag{1}$$

where $t \in (0, u]$,

$$\omega_n(u,t) = \int_u^t t_1^{\gamma_1 - 1}\, dt_1 \int_u^{t_1} t_2^{\gamma_2-\gamma_1-1}\, dt_2 \cdots \int_u^{t_{n-1}} t_n^{\gamma_n - \gamma_{n-1}-1}\, dt_n;$$

$$0 = \gamma_0 < \gamma_1 \leqslant \gamma_2 \leqslant \dots \to \infty, \qquad \sum_{\nu=1}^{\infty} \frac{1}{\gamma_\nu} = \infty;$$

$$a_0 = \varphi(u), \qquad a_k = u^{-\gamma_k + \gamma_{k-1}+1}\varphi_k(u), \qquad k = 1, 2, \dots;$$

$$\varphi_1(u) = \varphi'(u), \qquad \varphi_{k+1}(u) = \left(\frac{\varphi_k(u)}{u^{\gamma_k - \gamma_{k-1}-1}}\right)', \qquad k = 1, 2, \dots . \tag{2}$$

In the particular case when $\gamma_k = k,\ k = 0, 1, 2, \dots$, series (1) coincides with the ordinary Taylor series.

In the present paper theorem (2,7) of ([1]) is refined, and an analogous theorem is also proved for slow growth of the sequence $\{\gamma_\nu\}$—a case which was not considered in ([1]).

Definition 1. We shall agree to say that a function $\varphi(t)$ belongs to the class $C_{\{\dot{m}_n\}}([0,1])$ of analytic or quasi-analytic functions if the following conditions are satisfied:

1) $|\varphi^{(n)}(t)| \leqslant \dot{m}_n,\ n = 0, 1, 2, \dots,\ t \in [0, 1]$, where the sequence $\{\dot{m}_n\}$ is convexly regularized with respect to logarithms:

$$2) \quad \int^{\infty} \frac{\log \dot{T}(r)}{r^2}\, dr = \infty, \qquad \dot{T}(r) = \sup_{n \geqslant 1} \frac{r^n}{\dot{m}_n};$$

3) the counting function $n(r) = n(r, \beta)$ of the sequence

$$\left\{ \beta \frac{\dot{m}_n}{\dot{m}_{n-1}} \right\},$$

$0 < \beta < \infty$, is such that

$$\overline{\lim_{r \to \infty}} \frac{r^2}{n(r)} \int_r^{\infty} \frac{n(t)\, dt}{t(t^2 + r^2)} < \Delta_1 = \log \Delta. \tag{3}$$

Definition 2. We shall agree to say that

$$\varphi(t) \in C_{\{\dot{m}_n\},\varkappa}([0, 1]),$$

if

$$t^{\varkappa} \varphi(t) \in C_{\{\dot{m}_n\}}([0, 1]),$$

where $\varkappa$ is an arbitrary number.

Theorem 1. Every function $\varphi(t)$ belonging on $[0, 1]$ to some class $C_{\{\dot{m}_n\},\varkappa}$ of analytic or quasi-analytic functions is expanded into the quasipower series, convergent on $(0, u]$, $0 < u < 1$,

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n(u, t)$$

for the sequence $\{\gamma_\nu\}$, if the conditions

$$\gamma_0 = 0, \qquad \gamma_\nu = \beta \frac{\dot{m}_\nu}{\dot{m}_{\nu-1}}, \qquad \nu = 1, 2, \dots,$$

are satisfied, where $\beta > 6ea\Delta$, $a \geq 2\left(1 + \sqrt{\frac{1}{1-u}}\right)$, and Δ is defined in (3).

Corollary. Theorem 1 is valid, in particular, when condition (3) of Definition 1 is replaced by the condition

$$\sup_{t>r} \frac{n(t)}{t} < \frac{n(r)}{r} \log \Delta. \tag{3'}$$

Theorem (2,7) of work [1] is a special case of the corollary of Theorem 1, since from the condition imposed there on $\left\{\frac{\dot m_n}{\dot m_{n-1}}\right\}$ it follows that $\frac{n(t)}{t} \downarrow$ monotonically, and moreover the number β obtained here is smaller than the corresponding coefficient of $\frac{\dot m_n}{\dot m_{n-1}}$ in (1).

Let us note that Theorem 1 gives a solution of Carleman' s problem, posed in 1926, on representing a quasi-analytic function by its element (the set of successive derivatives at one point) under an additional restriction, namely under condition (3).

We now consider the case when the counting function $n(r)$ of the sequence $\left\{\frac{m_n}{m_{n-1}}\right\}$ is such that

$$n(r)\Delta \geq r, \tag{4}$$

where $\Delta > 0$ is some number. In the latter case we shall say that the sequence $\left\{\frac{m_n}{m_{n-1}}\right\}$ grows slowly.

Definition 1′. A function $\varphi(t)$ on $[0,1]$ belongs to the class $C_{\{m_n\}}([0,1])$ of analytic or quasi-analytic functions if the following conditions are satisfied:

1) $|\varphi^{(n)}(t)| \leq m_n, \quad n = 0,1,2,\dots, \ t \in [0,1]$, where the sequence $\{m_n\}$ is logarithmically convex regularized;
2)

$$\int^{\infty} \frac{\log T(r)}{r^2}\, dr = \infty, \qquad T(r) = \sup_{n\geq 1} \frac{r^n}{m_n}.$$

The class of functions $C_{\{\dot m_n\}}([0,1])$ differs from the class $C_{\{m_n\}}([0,1])$ in that in the latter case the third condition (3) is omitted.

Analogously, the class

$$C_{\{m_n,\varkappa\}}([0,1])$$

is defined.

Theorem 2. Every function

$$\varphi(t) \in C_{\{m_n,\varkappa\}}([0,1])$$

can be expanded into a quasi-power series convergent on $(0,u]$, $0 < u < 1$,

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n(u, t)$$

for the sequence $\{\gamma_\nu\}$, if the conditions

$$\gamma_0 = 0, \qquad \gamma_\nu = \beta \frac{m_\nu}{m_{\nu-1}}, \qquad \nu = 1, 2, \dots,$$

are satisfied, where $\beta > 12 \cdot 3^{\Delta} e$, and $\Delta > 0$ is defined in (4).

In contrast to the proof of theorem (2,7) of paper [1], here the function $\varphi(t)$ is first expanded in a Fourier series in Legendre polynomials, then generalizedly differentiated in the sense of (2), after which $|\varphi_{n+1}(t)|$ is estimated from above. The proof is completed by the estimate

$$R_n(u, t) = \varphi(t) - \sum_{k=0}^{n} a_k \omega_k(u, t) =$$

$$= \int_u^t \varphi_{n+1}(t_0)\, dt_0 \int_{t_0}^t t_1^{\gamma_1 - 1}\, dt_1 \int_{t_0}^{t_1} t_2^{\gamma_2 - \gamma_1 - 1}\, dt_2 \cdots \int_{t_0}^{t_{n-1}} t_n^{\gamma_n - \gamma_{n-1} - 1}\, dt_n.$$

Let us note that the estimate of $|\varphi_{n+1}(t)|$ under the conditions of the second theorem differs in many respects from the same estimate in theorem 1.

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Received
21 V 1960

REFERENCES

[1] G. V. Badalyan, *Izv. AN ArmSSR*, 6, No. 5–6 (1953).

Note: Figure translations are in progress. See original paper for figures.