SEMIGROUPS OF ONE-TO-ONE TRANSFORMATIONS

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Abstract

Full Text

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MATHEMATICS

1°. Let Σ_1 be the class of all semigroups with left cancellation and without left identities ([3]), and let Σ_2 be the class of all semigroups with left cancellation, with left invertibility ([2]), and without idempotents. Every semigroup of the class Σ_1 is isomorphic to a subsemigroup of the semigroup H_Ω of all one-to-one mappings of some infinite set Ω into itself, and every semigroup of the class Σ_2 is isomorphic to a subsemigroup of the semigroup P_Ω of all $a \in H_\Omega$ whose defect ([6]) is equal to the cardinality of Ω. Moreover, H_Ω belongs to the class Σ_1, and P_Ω to the class Σ_2. It follows from this that the study of the semigroups H_Ω and P_Ω is important not only from the point of view of transformation theory, but also from the point of view of the general theory of semigroups. In the present note all stable equivalences and normal complexes of the semigroups H_Ω and P_Ω are described. Owing to these results, a certain sufficient condition has been found for the possibility of embedding semigroups with left cancellation in semigroups that are simple with respect to homomorphisms. A number of properties of stable equivalences of the semigroups H_Ω and P_Ω are considered; in particular, the dependence of these equivalences on the stable equivalences of the symmetric groupoid ([1]) and of the symmetric generalized group ([5]) is established.

Terms of semigroup theory that are not defined are used in the usual sense ([2]). We shall denote the cardinality of a set Δ by $\tau\Delta$. Let a and b be mappings of the set Ω into itself, and let $\Delta \subset \Omega$. Denote by $a\Delta$ the image of Δ under a, and by $\Pi(a, b)$ the set of all such $\alpha \in \Omega$ that $a\alpha \neq b\alpha$, and let $\tau\Pi(a, b) = \omega(a, b)$.

2°. In what follows, by Ω we shall mean an infinite set, and by q an infinite cardinal number $\leq \lambda$, where λ is the least cardinal number $> \tau\Omega$. Let A_Ω be an arbitrary semigroup of transformations of the set Ω. Denote by $\sigma[q]$ the binary relation of the semigroup A_Ω such that elements $x, y \in A_\Omega$ are in the relation $\sigma[q]$ if and only if $\omega(x, y) < q$.

It is obvious that in any semigroup A_Ω the relation $\sigma[q]$ is a left-stable equivalence.

Theorem 1. *In order that in the semigroup A_Ω all relations $\sigma[q]$ be stable equivalences, it is necessary and sufficient that, for any $a, b, c \in A_\Omega$, $\Delta \subset \Omega$,* *from $c\Delta = \alpha \in \Pi(a,b)$ it follow that $\tau\Delta < r$, where r is the least infinite cardinal number $> \omega(a,b)$.*

Theorem 2. *All relations $\sigma[q]$ are stable equivalences of the semigroup P_Ω ([10]), and these relations exhaust all stable equivalences of the semigroup P_Ω distinct from equality.*

3°. Let $a \in P_\Omega$, $q \leq \lambda$ ([20]). Denote by $M(a,q)$ the set of all such $b \in P_\Omega$ that $\omega(a,b) < q$. From Theorem 2 the following theorems are derived:

Theorem 3. *Every set $M(a,q)$ is a normal complex of the semigroup P_Ω, and all normal complexes of the semigroup P_Ω consisting of more than one element are exhausted by the sets $M(a,q)$.*

Theorem 4. The semigroup P_Ω has no normal subsemigroups distinct from P_Ω.

4°. Let R be such a subset of P_Ω that if $a \in R$, $x \in P_\Omega$, $a\Omega \supset x\Omega$, $\tau(a\Omega \setminus x\Omega) = \tau\Omega$, then $x \in R$.

Theorem 5. Every subset of the form R of the semigroup P_Ω is its right ideal, and these ideals exhaust all ideals of the semigroup P_Ω.

5°. Let $\xi \leqslant \tau\Omega$, $\Delta \subset \Omega$, $\tau\Omega = \tau\Delta$. Denote by H_Ω^ξ and $H_{\Omega,\Delta}$, respectively, the sets of all such $x \in H_\Omega$ and $z \in H_\Omega$ (1°) that $\delta(x) \geqslant \xi$ and $z\Omega \subset \Omega$, where $\delta(x) = \tau(\Omega \setminus x\Omega)$.

Theorem 6. The only two-sided and right ideals of the semigroup H_Ω are, respectively, all the sets H_Ω^ξ and $H_{\Omega,\Delta}$. Moreover, every left ideal of H_Ω is its two-sided ideal.

6°. Let $a, b \in H_\Omega$, $a\Omega = b\Omega$. Denote by $[a,b]$ such a $c \in H_\Omega$ that $ca = b^{-1}aa$ for every $a \in \Omega$. Let S_Ω be the group of all $a \in H_\Omega$ for which $a\Omega = \Omega$ and $q \leqslant \lambda$ (2°).

Denote by R_q the set of all such $x \in S_\Omega$ that $\omega(x,e) < q$, where e is the identity in S_Ω, and by R_1 the alternating subgroup of the group S_Ω.

Definition. Let

$$0 \leqslant \xi_0 \leqslant \xi_1 \leqslant \xi_2 < \cdots < \xi_r = \lambda$$

be such a sequence of cardinal numbers (finite or infinite) that if ξ_1 is infinite, then $\xi_0 = \xi_1$, and if ξ_k ($k \geqslant 1$) is finite, then $\xi_k - \xi_{k-1} \leqslant 1$. To each ξ_k assign such an infinite cardinal number $\eta_k \leqslant \lambda$ that if $i < k$, then $\eta_i \leqslant \eta_k$. If ξ_{n_0} ($n_0 \geqslant 1$) is finite, and $\xi_{n_0+1} = \lambda_0$, where λ_0 is the cardinality of a countable set, then assign to the number n_0 a finite natural number $m_{n_0} \geqslant 1$. Let R_q be such that $q = \eta_0$ when $\eta_0 > \lambda_0$ and $q = 1, \lambda_0$ when $\eta_0 = \lambda_0$. Denote by $\sigma[\xi,\eta]$ such a binary relation of the semigroup H_Ω that transformations $a, b \in H_\Omega$ are in the relation $\sigma[\xi,\eta]$ if and only if one of the conditions is satisfied: 1) the transformations a and b are equal; 2) $a\Omega = b\Omega$, $\delta(a) = \xi_0$, $[a,b] \in R_q$,

$\omega(a,b) < \eta_0$; 3) $\xi_{n_0} \leqslant \delta(a), \delta(b) < \lambda_0$, $\omega(a,b) < \eta_{n_0}$, $\delta(a) - \delta(b) = m_{n_0} p$ ($p = 0, \pm 1, \pm 2, \ldots$); 4) $\xi_k \leqslant \delta(a), \delta(b) < \xi_{k+1}$, $\omega(a,b) < \eta_k$ ($k \neq 0,\ k \neq n_0$).

Theorem 7. Every relation $\sigma[\xi, \eta]$ is a stable equivalence of the semigroup H_Ω, and these equivalences exhaust all stable equivalences of the semigroup H_Ω.

7°. Let $\xi_1 < \xi_2 \leqslant \lambda$ be arbitrary cardinal numbers, $\eta \leqslant \lambda$ infinite, $a \in H_\Omega$, $\xi_1 \leqslant \delta(a) < \xi_2$. Denote by $M_1(a, \xi_1, \xi_2, \eta)$ the set of all such $x \in H_\Omega$ that $\xi_1 \leqslant \delta(x) < \xi_2$, $\omega(a,x) < \eta$. Let $n \geqslant 0$, $m \geqslant 1$ be finite, $n \leqslant \delta(a) < \lambda_0$. Denote by $M_2(a, n, m, \eta)$ the set of all such $x \in H_\Omega$ that $n \leqslant \delta(x) < \lambda_0$, $\omega(a,x) < \eta$, $\delta(a) - \delta(x) = mp$ ($p = 0, \pm 1, \pm 2, \ldots$). Further, let $\delta(a) = n$, R_q (6°) be such that $q = \eta$ if $\eta > \lambda_0$, and $q = 1, \eta$ if $\eta = \lambda_0$. Denote by $M_3(a, n, R_q, \eta)$ the set of all $x \in H_\Omega$ for which $x\Omega = a\Omega$, $[a,x] \in R_q$, $\omega(a,x) < \eta$.

Theorem 8. Every set $M_1(a, \xi_1, \xi_2, \eta)$, $M_2(a, n, m, \eta)$, $M_3(a, n, R_q, \eta)$ is a normal complex of the semigroup H_Ω, and these sets exhaust all normal complexes of the semigroup H_Ω containing more than one element.

8°. Let $\xi, \eta \leqslant \lambda$ be infinite. Denote by $R(\xi, \eta)$ the set of all such $a \in H_\Omega$ that $\delta(a) < \xi$, $\omega(a,e) < \eta$, and if $\xi = n \geqslant 0$ is finite, then by $R_1(n, \eta)$ the set of all $b \in H_\Omega$ for which $\delta(b) = kn$, $\omega(b,e) < \eta$, where $k = 0, \pm 1, \pm 2, \ldots$.

Theorem 9. The only normal subsemigroups of the semigroup H_Ω are the unit subsemigroup, the alternating subgroup of the group S_Ω, and all subsets of the form $R(\xi, \eta)$ and $R_1(n, \eta)$.

From Theorem 9 there follows the following known ([8]) corollary:

Corollary. The only normal divisors of the group S_Ω are the unit subgroup and all R_ξ, where $\xi = 1$, $\xi_0 \leqslant \xi \leqslant \lambda$.

9°. A semigroup is called **simple** if each of its stable equivalences is either equality or contains all pairs of elements of this semigroup. In ([4], [7]) three examples were first constructed of simple semigroups containing elements of infinite order and distinct from groups. The following theorem gives one more example of such semigroups.

Theorem 10. *The factor semigroup of the semigroup P_Ω by the equivalence $\sigma[\tau\Omega]$ (2°) is a simple semigroup with left reversibility and without idempotents.*

10°. Let A be an infinite semigroup with left cancellation, let $a, b \in A$, and let Δ be the set of all such $x \in A$ that $ax \neq bx$. If for any a and b one always has $\tau\Delta = \tau(A \setminus aA)$, then the semigroup A will be called an l-semigroup. From Theorem 10 the following theorem is derived:

Theorem 11. *Every l-semigroup is a subsemigroup of some simple semigroup with left reversibility and without idempotents.*

Let F be the semigroup of such continuous strictly monotone functions $f(x)$, defined on the interval $[a,b]$, $a < b$, that $f[a,b] \neq [a,b]$. From Theorem 11 the following corollary follows:

Corollary. *Each of the semigroups P_Ω, F is a subsemigroup of some simple semigroup with left reversibility and without idempotents.*

Since every semigroup of class Σ_2 is isomorphic to a subsemigroup of some semigroup P_Ω, it follows from this:

Corollary. *Every semigroup with left reversibility, with left cancellation and without idempotents, is a subsemigroup of some simple semigroup with left reversibility and without idempotents.*

11°. Let A_1 be a subsemigroup of the semigroup A_2. An equivalence σ_1 of the semigroup A_1 is called the restriction of an equivalence σ_2 of the semigroup A_2 if elements a and b of the semigroup A_1 are in the relation σ_1 if and only if they are in A_2 in the relation σ_2. Let C_Ω be the symmetric groupoid ([1]) and W_Ω the symmetric generalized group ([5]). From ([1], [5]), thanks to Theorems 2 and 7, the following theorems are derived:

Theorem 12. *Every stable equivalence of the semigroup P_Ω is the restriction of some stable equivalence of each of the semigroups C_Ω, W_Ω, and H_Ω.*

Theorem 13. *All stable equivalences of the semigroup H_Ω that are restrictions of stable equivalences of each of the semigroups C_Ω, W_Ω, are exhausted by all equivalences of the form $\sigma[q]$ (2°).*

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Note: Figure translations are in progress. See original paper for figures.