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Abstract

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THE BOUNDARY VALUE PROBLEM FOR LINEAR AND QUASILINEAR PARABOLIC EQUATIONS

(Presented by Academician V. I. Smirnov on 17 III 1961)

Parabolic equations are considered in the cylinder $Q_T = \Omega \times [0, T]$:

$$u_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) u_{x_j} + b_i u + e_i \right) + \sum_{i=1}^{n} c_i u_{x_i} + du + f = 0; \tag{1}$$

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[a_i(x,t,u,u_{x_k}) \right] + a(x,t,u,u_{x_k}) = 0 \tag{2}$$

with

$$u\big|_{\Gamma} = \varphi(x,t) \tag{3}$$

where S is the boundary of the domain Ω, $\Gamma = \{S \times [0,T]\} \cup \{\Omega, t = 0\}$.

Numerous works are devoted to the study of linear parabolic equations. The sharpest result concerning estimates of their solutions was obtained in Nash’ s paper (^{1}). Namely, Nash gave an estimate in the norm $C_{0,\alpha}$ of the fundamental solution of the equation

$$u_t = \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) u_{x_j} \right)$$

in terms of $\max |a_{ij}|$ and the ellipticity constant. Among works on quasilinear parabolic equations with many independent variables, the strongest nonlinearities were considered in (^{2}).

In the present work we have succeeded for the first time in proving solvability “in the large” of problem (2), (3) under such conditions on a_i and a which, in a certain sense, are necessary (see (^{3},^{4})), and also in giving an estimate in the norm $C_{0,\alpha}(Q_T)^*$ of generalized solutions of the boundary value problem for linear parabolic equations under very small assumptions on the coefficients.

We mainly deal with a priori estimates of the solutions u and of their derivatives for equations (1) and (2) in Hölder norms $C_{0,\alpha}$. It is known that generalized solutions of these equations satisfy a certain integral identity. In addition, as is not difficult to show, for solutions of (1) the inequality

$$\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} |u-k|^2\xi^2(x)\,dx + \nu\int_{A_{k,\rho}(t)} |\nabla u|^2\xi^2\,dx \leq$$

$$\leq \frac{\gamma}{\delta^2\rho^2}\int_{A_{k,\rho}(t)} |u-k|^2\,dx + \gamma\,\mathrm{mes}^{1-2/q} A_{k,\rho}(t) \tag{4}$$

holds, as well as the analogous inequality (4′) for $B_{k,\rho}(t)$. Here and below $K(\rho)$ denotes an arbitrary ball of radius ρ; $A_{k,\rho}(t)$ is the set

* The norm of $u(x,t)$ in $C_{0,\alpha}(Q_T)$ is defined by the equality

$$|u|_{C_{0,\alpha}(Q_T)} \equiv \max_{(x,t)\in Q_T} |u(x,t)| + \max_{(x,t),(x',t')\in Q_T} \frac{|u(x,t)-u(x',t')|}{(|x-x'|^2+|t-t'|)^{\alpha/2}}.$$

points of $K(\rho)\cap\Omega$, where $u(x,t)>k$; $B_{k,\rho}(t)$ is the set of points of $K(\rho)\cap\Omega$ where $u(x,t)<k$. The function $\xi(x)$ is continuously differentiable, nonnegative, equal to one in $K(\rho-\sigma\rho)$ and to zero outside $K(\rho)$. The constant

$$\nu = 2\min_{(x,t)\in Q_T}\frac{a_{ij}\xi_i\xi_j}{\sum_i \xi_i^2}$$

is here and below assumed to be positive; γ is determined by $M = \mathrm{vrai}\max_{Q_T}|u|$, and also by

$$\max_{Q_T}|a_{ij}|, \qquad \max_{0\leq t\leq T}(\|b_i,c_i,e_i\|_{L_q(\Omega)}, \|d,f\|_{L_{q/2}(\Omega)}), \qquad q>n. \tag{5}$$

The number k in (4) is arbitrary for $K(\rho)\subset\Omega$ and is larger than the maximum of u on the intersection of $K(\rho)$ with S, if the latter is nonempty.

With respect to the functions $a_i(x,t,u,p_k)$ and $a(x,t,u,p_k)$ we shall assume that the conditions

$$a_i(x,t,u,p_k)p_i \geq \nu_0(|u|)p^2 - \mu(|u|); \qquad p = \left(\sum_{i=1}^n p_i^2\right)^{1/2}; \qquad \nu_0>0;$$

$$\sum_i |a_i(x,t,u,p_k)|\,(p+1)|a(x,t,u,p_k)| \le \mu(|u|)(p^2+1); \qquad \mu > 0 \tag{6}$$

are satisfied.

Then inequalities (4) and (4′) are also valid for solutions of equation (2), but not for all k, rather for those for which $\operatorname{mes} A_{k+\delta,\rho}(t) = 0$ (respectively $\operatorname{mes} B_{k-\delta,\rho}(t) = 0$), where δ is a positive number determined by the constants $\nu_0(M)$, $\mu(M)$ from conditions (6).

Denote by $\mathfrak{B}$ the set of functions satisfying inequalities (4) and (4′) for only the indicated k and for all $t \in [0,T]$.

Theorem 1. The functions $u(x,t)$ of the class $\mathfrak{B}$ satisfy the Hölder condition

$$|u(x,t) - u(x',t')| \le C d^{-\alpha}(x,t)\big[|x-x'|^2 + |t-t'|\big]^{\alpha/2}. \tag{7}$$

with positive constants C and α, determined only by M, γ, δ, and ν. Here $t \ge t'$, $d(x,t)$ is the smaller of $\sqrt{t}$ and the distance from x to S.

We shall say that S satisfies condition A if there exist numbers $a_0 < 1$ and A_0 such that, for any sphere $K(\rho)$ with center on S of radius $\rho \le A_0$,

$$\operatorname{mes}(K(\rho) \cap \Omega) \le a_0 \operatorname{mes} K(\rho).$$

Denote by Γ the set of points lying on the lateral surface and the lower base of Q_T.

Theorem 2. Let S satisfy condition A. Then any function u from $\mathfrak{B}$, whose boundary values on Γ satisfy the Hölder condition with some exponent $\beta < 0$, satisfies in $\overline{Q}_T$ the Hölder condition

$$|u(x,t) - u(x',t')| \le C\big[|x-x'|^2 + |t-t'|\big]^{\alpha/2}$$

with certain positive constants α and C, determined by $M, \gamma, \delta, \nu, a_0, A_0, |u|_{C_{0,\beta}(\Gamma)}$.

Theorems 1 and 2 give, for solutions of equations (1) and (2), an estimate of $|u|_{C_{0,\alpha}}$ in terms of $M = \operatorname{vrai\,max}_{Q_T} |u|$. An estimate of M for solutions of equation (1) is given by the theorem:

Theorem 3. For a solution u of equation (1), $M = \operatorname{vrai\,max}_{Q_T} |u|$ is estimated in terms of $\max_\Gamma |u|$ and the constants from (5).

With the aid of the a priori estimates guaranteed by Theorems 1-3, various existence theorems are established by known methods, for example, the following:

Theorem 4. Let the constants in (5) be finite and let S satisfy condition A. Then there exists a unique generalized solution u of problem (1), (3), belonging to $C_{0,\alpha}(\overline{Q}_T)$, $\alpha > 0$, with $u_{x_i} \in L_2(Q_T)$, if and only if $u(x,0) \in C_{0,\beta}(\overline{\Omega})$, and $u|_S = 0$.

Suppose that $a_i(x,t,u,p_k)$ and $a(x,t,u,p_k)$, as functions of their arguments, belong to the classes $C_{1,\alpha}$ and $C_{0,\alpha}$, $\alpha > 0$. Suppose that the inequalities (6) are satisfied for them, so that the order of growth of a_i with respect to p is equal to 1, and the order of growth of a with respect to p is not greater than 2. Suppose that this order of a_i, when a_i is differentiated with respect to p_k, decreases by at least 1, and when differentiated with respect to u and x_k does not increase. Finally, suppose that the ellipticity condition is fulfilled in the form

$$\nu(|u|)\sum_{i=1}^{n}\xi_i^2 \leq \frac{\partial a_i(x,t,u,p_k)}{\partial p_i}\,\xi_i\xi_j \leq \mu(|u|)\sum_{i=1}^{n}\xi_i^2, \qquad \nu > 0.$$

Under these conditions the following is true.

Theorem 5. The norm of u_{x_i} in $C_{0,\alpha}(Q'_T)$ and $\max_{Q_T}|u_{x_i}|$ for solutions of equation (2) are estimated in terms of the data of the problem and M, if $S \in C_2$ and $u|_\Gamma \in C_1$.

Here and below $Q'_T = \Omega' \times [0,T]$, where Ω' is an interior subdomain of Ω.

From this theorem and Friedman' s theorem (^{5}) follows the possibility of estimating, for solutions u of equation (2), the norms of u_t and $u_{x_ix_j}$ in $C_{0,\alpha}(Q'_T)$ in terms of the data of the problem.

Theorem 6. Problem (2), (3) has a unique solution u from $C_{0,\alpha}(\overline{Q}_T)$, with $u_t, D_x^2 u$ from $C_{0,\alpha}(Q'_T)$ and $\max|u_{x_i}| < \infty$, if $u(x,0) \in C_{2,\alpha}(\Omega)$, $u|_\Gamma \in C_1$, $S \in C_2$, and if, with respect to a_i and a, the conditions of Theorem 5 and the inequality

$$\frac{\partial}{\partial u}\left[-\frac{\partial a_i(x,t,u,0)}{\partial x_i} + a(x,t,u,0)\right] \geq \text{const} > -\infty$$

are satisfied for all $(x,t) \in \overline{Q}_T$ and arbitrary u.

Theorems 1 and 2 are based on the following lemmas, valid for arbitrary functions from $\mathfrak{B}$.

Lemma 1. Let $K(\rho) \subset \Omega$ and $\operatorname{mes} A_{k,\rho}(t_0) \leq a\chi_n\rho^n$, where $\chi_n = \operatorname{mes} K(1)$. Then for any $a < 1$ there exist numbers β from $(0,1)$, $h < 1$, and $\chi > 0$, depending only on a and on the constants γ, δ, ν from (4), such that either

$$H = \max|u(x,t) - k| \leq \rho^{1-n/q} \quad \text{for } x \in A_{k,\rho}(t), \quad t \in [t_0, t_0 + \chi\rho^2],$$

or

$$\max_{t\in[t_0,t_0+\chi\rho^2]} \operatorname{mes} A_{k+\beta H,\rho}(t) \le h\chi_n\rho^n.$$

We shall denote by $Q(\rho)$ cylinders of the form $K(\rho)\times[t_0,t_0+\chi\rho^2]$. Take one of these cylinders and denote the oscillation of u in it by ω, the oscillation of u on the intersection of $Q(\rho)$ with Γ (if it exists) by ω_Γ, and

$$\max_{Q(\rho)} u = \mu_1, \qquad \min_{Q(\rho)} u = \mu_2.$$

Lemma 2. Let $\omega \le 4\omega_\Gamma$. Then for every $\theta > 0$ there exists an $s \ge 2$ such that either

1) $\omega \le 2^s\rho^\varepsilon$, where $\varepsilon = \min\{\beta, 1-n/q\}$,

or

2) $\operatorname{mes} A_{\mu_1-\omega/2^{s+1},\,\rho/4}(t) \le \theta\rho^n$, for $t\in[t_0+\frac{3}{4}\chi\rho^2,\ t_0+\chi\rho^2]$,

or

3) $\operatorname{mes} B_{\mu_2+\omega/2^{s+1},\,\rho/4}(t) \le \theta\rho^n$.

If $t_0 \le 0$, then in 2) and 3) t may be taken from the interval $[0, t_0+\chi\rho^2]$.

Lemma 3. Let

$$H = \max_{\substack{x\in A_{k,\rho}(t)\\ t\in[t_0,t_0+\Delta]}} |u(x,t)-k| \le \delta, \qquad \Delta \ge \chi\rho^2,$$

Moreover, for $K(\rho)$ intersecting S, we consider only those k which exceed the greatest value of u on the intersection of $K(\rho)\times[t_0,t_0+\Delta]$ with Γ. Then for every $\mu > 0$ there exists a $\theta > 0$ such that if $H \geqslant \rho^\varepsilon$, $\varepsilon = \min\{\beta, 1-n/q\}$, and $\max\limits_{t\in[t_0,t_0+\Delta]} \operatorname{mes} A_{k,\rho}(t) \leqslant \theta\rho^n$, then $\operatorname{mes} A_{k+H/2,\rho/2}(t) = 0$ for $t\in[t_0+\mu\rho^2, t_0+\Delta]$.

If $t_0 \leqslant 0$ and $\operatorname{mes} A_{k,\rho}(0) = 0$, then the latter assertion is valid for $t\in[0,t_0+\Delta]$.

Assertions analogous to Lemmas 1 and 3 are also true for the sets $B_{k,\rho}(t)$. We have formulated all the results under the assumption $n \geqslant 2$. For $n = 1$ many of them take a simpler form. For problem (2), (3) the existence of a classical solution $u(x,t)$ with u_t and u_{xx} in $C_{0,\alpha}(\overline{Q}_T)$ has been proved.

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Note: Figure translations are in progress. See original paper for figures.