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Abstract

Full Text

MATHEMATICS

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A CONDITION FOR THE EXPANDABILITY OF FUNCTIONS IN A QUASI-POWER SERIES OUTSIDE THE FUNDAMENTAL INTERVAL

(Presented by Academician M. V. Keldysh on 19 VII 1960)

In the paper ⁽¹⁾ a condition was established for the expandability of functions in the quasi-power series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right), \tag{1}$$

where $t \in (0, u]$, $\omega_0 \left(\frac{t}{u}, \gamma \right) = 1$,

$$\omega_n \left(\frac{t}{u}, \gamma \right) = \prod_{\nu=1}^n \gamma_{\nu} \cdot u^{-\gamma_n} \int_u^t t_1^{\gamma_1-1} dt_1 \int_u^{t_1} t_2^{\gamma_2-\gamma_1-1} dt_2 \dots \int_u^{t_{n-1}} t_n^{\gamma_n-\gamma_{n-1}-1} dt_n; \tag{2}$$

$$a_0 = \varphi(u), \quad a_n = \frac{u^{\gamma_{n-1}+1} \varphi_n(u)}{\prod_{\nu=1}^n \gamma_{\nu}}, \quad \nu = 1, 2, \dots;$$

$$\varphi_1(u) = \varphi'(u), \quad \varphi_{k+1}(u) = \left(\frac{\varphi_k(u)}{u^{\gamma_k-\gamma_{k-1}-1}} \right)', \quad k = 1, 2, \dots; \tag{3}$$

$$0 = \gamma_0 < \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \leq \dots, \quad \sum_{\nu=1}^{\infty} \frac{1}{\gamma_{\nu}} = \infty.$$

In the present paper we consider the question of expanding functions in the convergent series (1), but only for values $t > u$; we call this interval the **non-fundamental interval**.

Definition. We shall agree to say that

$$\varphi(t) \in C_{\{\prod_1 \gamma_{\nu}\}}^*([u, A]), \quad 0 < u < A < \infty,$$

if $\varphi(t)$ is infinitely differentiable and

$$|\varphi_n(t)| \leq C \prod_{\nu=1}^n \gamma_\nu t^{-\gamma_\nu - 1}, \quad t \in [u, A]. \quad (4)$$

Here $\varphi_n(t)$ is defined in (3), and $C > 0$ is an absolute constant.

It turns out that the conditions for the expandability of functions in the series (1) on the fundamental interval $(0, u]$ differ essentially from the conditions for expandability outside the fundamental interval, as follows from the following theorems.

Theorem 1. For every function

$$\varphi(t) \in C_{\{\prod_1 \gamma_\nu\}}^*([u, A]), \quad t \in [u, A], \quad 0 < u < A,$$

the inequality

$$|R_n(u, t)| \leq c \omega_n \left(\frac{t}{u}, \gamma \right), \quad (5)$$

holds.

where

$$R_n(u, t) = \varphi(t) - \sum_{k=0}^{n-1} a_k \omega_k \left(\frac{t}{u}, \gamma \right). \quad (6)$$

Consider sequences of numbers

$$\begin{aligned} 0 &= \alpha_0 < \alpha_1 < \alpha_2 < \dots, \\ 0 &= \beta_0 < \beta_1 < \beta_2 < \dots, \quad \beta_\nu \leq \alpha_\nu, \quad \nu = 1, 2, \dots \end{aligned} \quad (7)$$

and construct the functions $\omega_n \left(\frac{t}{u}, \alpha \right), \omega_n \left(\frac{t}{u}, \beta \right)$.

Theorem 2. For $t > u$ the inequalities

$$\prod_{\nu=1}^n \frac{\alpha_\nu}{\beta_\nu} G_n(1) \omega_n \left(\frac{t}{u}, \beta \right) \leq \omega_n \left(\frac{t}{u}, \alpha \right) \leq \prod_{\nu=1}^n \frac{\alpha_\nu}{\beta_\nu} G_n \left(\frac{t}{u} \right) \left(\frac{t}{u} \right)^{-\beta_n} \omega_n \left(\frac{t}{u}, \beta \right), \quad (8)$$

hold, where $\{\alpha_\nu\}, \{\beta_\nu\}$ are defined in (7),

$$G_n(x) = \frac{1}{2\pi i} \int_C \prod_{\nu=1}^{n-1} \frac{z + \beta_\nu}{z + \alpha_\nu} \frac{x^{-\zeta} d\zeta}{z + \alpha_n},$$

and the simple contour C encloses neighborhoods of the zeros of the denominator of the integrand.

Corollary. Under the conditions of Theorem 2 the inequalities

$$\prod_{\nu=1}^n \frac{\alpha_\nu}{\beta_\nu} \omega_n \left(\frac{t}{u}, \beta \right) \leq \omega_n \left(\frac{t}{u}, \alpha \right) \leq \prod_{\nu=1}^n \frac{\alpha_\nu}{\beta_\nu} \prod_{\nu=1}^{n-1} \frac{\alpha_n - \beta_\nu}{\alpha_n - \alpha_\nu} \left(\frac{t}{u} \right)^{\alpha_n - \beta_n} \omega_n \left(\frac{t}{u}, \beta \right). \quad (9)$$

With the aid of inequalities (5) and (9) one proves:

Theorem A. Let

$$\varphi(t) \in C_{\{\prod_1^n \gamma_\nu\}}^*([u, v]),$$

$$0 = \gamma_0 < \gamma_1 < \gamma_2 < \dots, \quad \gamma_\nu \leq \omega\nu, \quad \omega > 0,$$

$$\prod_{\nu=1}^n \frac{\gamma_\nu}{\omega\nu} \left[\left(\frac{v}{u} \right)^\omega - 1 \right]^n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (10)$$

Then the series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right)$$

converges uniformly for all $t \in [u, v]$.

Theorem B. Let

$$\varphi(t) \in C_{\{\prod_1^n \gamma_\nu\}}^*([u, v]),$$

$$\gamma_0 = 0, \quad \gamma_\nu \geq \omega\nu, \quad \omega > 0,$$

$$\lim_{n \rightarrow \infty} \prod_{\nu=1}^n \frac{\gamma_\nu}{\omega\nu} \prod_{\nu=1}^{n-1} \frac{\gamma_n - \omega\nu}{\gamma_n - \gamma_\nu} \left(\frac{v}{u} \right)^{\gamma_n - \omega n} \left[\left(\frac{v}{u} \right)^\omega - 1 \right]^n = 0. \quad (11)$$

Then the series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right)$$

converges uniformly on the whole interval $[u, v]$.

Remark. Theorems A and B show that, as the rate of growth of the sequence $\{\gamma_\nu\}$ increases, the interval of convergence $[u, v]$ of the series (1) generally narrows, and there is reason to suppose that, for sufficiently rapid growth of $\{\gamma_\nu\}$, the series (1) can converge only at the point $t = u$.

That this is indeed so, and that the sufficient conditions given in Theorems A and B cannot be substantially sharpened (in the sense of describing the phenomenon), is proved by the following propositions.

Theorem A₁. Under the conditions of Theorem A there exists a function $\varphi(t)$ for which, when

$$\frac{1}{n} \prod_{\nu=1}^{n-1} \frac{|\mu - \gamma_\nu|}{\omega n - \gamma_\nu} \left(\frac{v_1}{u} \right)^{\omega \nu} \left[\left(\frac{v_1}{u} \right)^\omega - 1 \right]^n > c > 0, \quad (12)$$

where $u < v_1 < v$, μ is a certain number, the series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right)$$

diverges for all $t \in [v_1, v]$.

Theorem B₁. Under the conditions of Theorem B there exists a function $\varphi(t)$ such that, when

$$\gamma_0 = 0, \quad \gamma_\nu > \omega(\nu + 1), \quad \omega > 0, \quad \nu = 1, 2, \dots;$$

$$\frac{1}{n} \prod_{\nu=1}^{n-1} \frac{|\mu - \gamma_\nu|}{\omega \nu} \left[\left(\frac{v_1}{u} \right)^\omega - 1 \right]^n > c > 0$$

the series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right)$$

diverges for all $t \in [v_1, v]$.

Theorem B'_1 . Under the conditions of Theorem B, if

$$\lim_{\nu \rightarrow \infty} \frac{\gamma_\nu}{\nu} = \infty,$$

there exists a function $\varphi(t)$ such that the series

$$\varphi(t) = \sum_{n=0}^{\infty} a_n \omega_n \left(\frac{t}{u}, \gamma \right)$$

diverges for all $t > u$.

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References

1. G. V. Badalyan, *Izv. AN ArmSSR*, **6**, No. 5–6 (1953).

Note: Figure translations are in progress. See original paper for figures.

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