ON THE COMPUTATION OF THE INDEX OF AN ISOLATED FIXED POINT OF A COMPLETELY CONTINUOUS VECTOR FIELD

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Abstract

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ON THE COMPUTATION OF THE INDEX OF AN ISOLATED FIXED POINT OF A COMPLETELY CONTINUOUS VECTOR FIELD

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Knowledge of the topological index of a fixed point of a completely continuous vector field (see (^{1})) makes it possible to solve, in a relatively simple way, a number of problems of nonlinear analysis. We have in mind here various existence theorems, uniqueness and non-uniqueness theorems for solutions, estimates and the exact determination of the number of solutions, the study of convergence of certain approximate methods, the study of bifurcation points, and so on. At the same time, the index is computed simply (see (^{1,2})) only in the case when the linearized field is nondegenerate. In (^{1,3}) particular classes of fields with a degenerate linear part are indicated, for which the index is computed by simple formulas.

Below, N denotes the order of the zero eigenvalue of the vector field linearized at the fixed point.

The article proposes a fairly simple and, as it seems to the authors, sufficiently general algorithm for reducing the problem of computing the index to the computation of the rotation of a certain field on a sphere of dimension $N-1$. In the case most important for applications, when the linear field has a one-dimensional degeneracy, the computation of the index is reduced by this algorithm to determining the sign of the first nonzero number in a sequence of numbers determined by a certain recurrent process (analogously to the way in which, in the study of functions for an extremum, one has to determine the sign of the corresponding derivatives).

The proposed algorithm is also useful in the study of vector fields in finite-dimensional spaces.

1. Let us first consider a particular class of vector fields $\Phi_0 = I - A_0$, where I is the identity operator and A_0 is a completely continuous operator acting in a real Banach space E and admitting the representation

$$A_0 = P_0 + C_2 + \cdots + C_m + D, \qquad (1)$$

in which P_0 is the operator of projection onto an N-dimensional subspace E_0;

each of the operators C_i is a homogeneous operator of order i; D is an operator containing terms of order higher than m.

Let

$$P_0C_2x \equiv P_0C_3x \equiv \cdots \equiv P_0C_{k-1}x \equiv 0 \quad (x \in E) \tag{2}$$

and $P_0C_kx \not\equiv 0$. In addition, let

$$(I - P_0)C_2P_0x \equiv \cdots \equiv (I - P_0)C_{sP}0x \equiv 0 \quad (x \in E) \tag{3}$$

and $(I-P_0)C_{s+1}P_0x \not\equiv 0$. The number $s+K$ will be called the P_0-characteristic of the vector field $I - A_0$ at the zero point.

By the order of degeneracy of the field $I - A_0$ at the zero point we shall mean the least natural number r such that

$$\|P_0C_2P_0x + \cdots + P_0C_{rP}0x\| \geq \alpha\|P_0x\|^2 \quad (\alpha > 0,\ \|x\| \leq \rho_0). \tag{4}$$

In the case of one-dimensional E_0, condition (4) takes the especially simple form:

$$f(C_re_0) \neq 0, \tag{5}$$

where $e_0 \in E_0$ and $\|e_0\| = 1$, while f is a linear functional satisfying the conditions $f(x) = f(P_0x)$, $f(e_0) > 0$. In the case of one-dimensional E_0, conditions (2) and (3) also take a simpler form; the first of them is written as

$$f(C_2x) \equiv f(C_3x) \equiv \ldots \equiv f(C_{k-1}x) \equiv 0 \quad (x \in E), \tag{6}$$

and the second in the form

$$f(C_ie_0)e_0 = C_ie_0 \quad (i = 2, \ldots, s). \tag{7}$$

Lemma 1. *If the order of degeneracy is less than the P_0-characteristic, then the zero fixed point is isolated and its index γ_0 is equal to the rotation γ^* of the vector field*

$$\psi_x = -P_0C_2x - P_0C_3x - \ldots - PC_{rx} \tag{8}$$

on the sphere $\|x\| = \rho$ $(\rho \leq \rho_0)$ of the finite-dimensional space E_0.

In the case of one-dimensional E_0, the rotation γ^* is equal to zero if r is even, and if r is odd, then γ^* is determined by the sign of $f(C_re_0)$.

2. In this section we shall show how the study of a fixed point (which below, without loss of generality, we consider to be zero) of an arbitrary completely continuous vector field $1-A$ can be reduced to the study of a field $I-A_0$ with an operator A_0 admitting the representation (1). In other words, we shall show how to pass to a field $I-A_0$ with an operator A_0, whose derivative $A_0'(\theta)$ at the zero point θ is a projection operator.

Let B be the Fréchet derivative of the operator A at the point θ; the operator B is a linear completely continuous operator (^{1}). Let 1 be an eigenvalue of the operator B of order N; by the order of an eigenvalue we mean the dimension of the subspace E_0 of eigenvectors (associated vectors play no role). Denote by B_1 such a linear completely continuous operator that

$$(I-B_1)^{-1}(B-B_1)=P_0, \tag{9}$$

where P_0 is some linear projection operator onto the subspace E_0 of eigenvectors. Naturally, it is assumed here that the operator $1-B_1$ is continuously invertible.

Put

$$A_0=(I-B_1)^{-1}(A-B_1). \tag{10}$$

From the equality

$$I-A_0=(I-B_1)^{-1}(I-A) \tag{11}$$

it follows that the zero point will be an isolated fixed point of the field $I-A$ if and only if it is an isolated fixed point of the field $I-A_0$. From the theorem on the product of rotations (^{1}) it follows that the indices γ and γ_0 of the zero fixed point of the fields $I-A$ and $I-A_0$ are related by the equality

$$\gamma=(-1)^{\beta_0}\gamma_0, \tag{12}$$

where β_0 is the sum of the multiplicities of the positive eigenvalues of the operator B_1 that are greater than 1 (by the multiplicity of an eigenvalue we mean the dimension of the root subspace consisting of all solutions of all equations $(I-B_1)^p x=\theta$, where $p=1,2,\ldots$).

From equalities (9) and (10) it follows that $A_0'(\theta)=P_0$.

For the construction of the operator B_1, one may apply the usual Lyapunov-Schmidt construction.

Let $e_1,\ldots,e_N$ be a complete system of linearly independent eigenvectors of the operator B corresponding to the eigenvalue equal to 1, and let $\psi_1,\ldots,\psi_N$ be such linear functionals that $\psi_i(e_j)=\delta_{ij}$. Let $\varphi_1,\ldots,\varphi_N$ be linearly independent eigenfunctionals of the adjoint operator B^*, corresponding to the same eigenvalue, and let $g_1, \dots, g_N$ be such linearly independent elements that $\varphi_i(g_j) = \delta_{ij}$. Then the operator B_1 may be defined by the equality

$$B_1 x = Bx \sum_{i=1}^{N} \psi_i(x) g_i. \tag{13}$$

3. If, after passing to the field $I - A_0$, its P_0-characteristic is greater than the order of degeneracy, then the zero fixed point of the field $I - A$ is isolated, and its index is computed with the aid of Lemma 1 and formula (12).

Let now the P_0-characteristic not exceed the order of degeneracy. In this case we pass from the field $I - A_0$ to the field $I - A_1$, where

$$A_1 = A_0(I + F) - F, \tag{14}$$

$$F = (I - P_0) C_{s+1} P_0. \tag{15}$$

It can be verified that zero θ is an isolated fixed point of the field $I - A_0$ if and only if it is an isolated fixed point of the field $I - A_1$. The indices of the zero fixed point of these two fields will be the same.

A calculation shows that the operator A_1 admits the same representation (1) as the operator A_0 (in this representation the first s terms are preserved without change). Fundamental for us is the fact expressed by Lemma 2.

Lemma 2. *The P_0-characteristic of the field $I - A_1$ is greater than the P_0-characteristic of the field $I - A_0$.*

We now compute the order of degeneracy at zero θ of the field $I - A_1$; if it is smaller than the P_0-characteristic, then the investigation of the fixed point is complete.

If the order of degeneracy is greater than the P_0-characteristic or equal to it, then one must again pass to a new vector field by formulas (14)–(15), etc.

4. The scheme described above is the proposed algorithm. The number of possible steps (connected with the replacements (14)–(15)) is determined by the smoothness of the operator A. Let us note an essential drawback of the algorithm: if the fixed point is not isolated, then no finite number of passages to new fields with a larger P_0-characteristic will reveal this circumstance. The algorithm is convenient if the fixed point is isolated and its index is determined by a finite number of terms in the expansion of the operator by Taylor's formula. This number of terms also determines the number of passages to new vector fields (by formulas (14)–(15)) after which Lemma 1 may be applied.

5. We give one example. Consider the equation

$$x(t) = \lambda \int_0^\pi H(t,s)\, e^{3Cx(s)} \frac{\sin \nu x(s)}{\nu}\, ds, \tag{16}$$

where

$$H(t,s) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt \sin ns}{n}, \tag{17}$$

and the linear operator C is defined by the equality

$$C\left(\sum_n \xi_n \sin nt\right) = \sum_n \xi_n \cos nt. \tag{18}$$

Equation (16) for $v = 1$ describes the steepness of a wave on the surface of a heavy fluid; λ is a parameter determined by the characteristic velocity of the flow and the wavelength ([4,5]).

Davis in ([4]), as a model, considered equation (16) for $v = 3$ and found that this model equation, for the values $\lambda = 1, 2, \ldots$, has continua of solutions (Davis found these solutions in explicit form).

The investigation of equation (16) can be carried out by topological methods. In this equation make the substitution

$$x(t) = Hy(t), \qquad Hy(t) = \int_0^\pi H(t,s) y(s)\, ds. \tag{19}$$

Then, to determine $y(t)$, we obtain the equation

$$y(t) = \lambda e^{3CHy(t)} \frac{\sin vHy(t)}{v}, \tag{20}$$

where the operator λA, defined by the right-hand side, will be completely continuous as an operator acting from L_2 into C. From the general theorems on bifurcation points it follows that all the numbers $\lambda = 1, 2, \ldots$ will be bifurcation points for the equation $y = \lambda A y$.

Let λ be equal to the natural number n_0. From Davis' s result it follows that zero will not be an isolated fixed point of the field $I - n_0 A$, if $v = 3$. For $v \neq 3$, zero, as it turns out, is an isolated solution, and its index γ is determined by the equality

$$\gamma = (-1)^{n_0 - 1} \operatorname{sign}(v^2 - 9) \tag{21}$$

(for the proof it is sufficient to apply the algorithm described, using the passage by formulas (14)–(15) once). From (21) it is easy to draw a conclusion (see in (^{2}) the general theorems on bifurcation points) as to for what values of λ, close to n_0, equation (16) has nonzero solutions. Davis' s equation ($v = 3$) and the exact equation ($v = 1$) provide an interesting example in which a change, in the equations, of terms of the third order of smallness while the nonzero terms of the first and second orders of smallness are unchanged leads to a qualitatively different picture: in one equation nonzero solutions exist for discrete values of the parameter, while in the other they exist for values of the parameter filling intervals.

We note that a number of delicate nonlocal theorems on solutions of equation (16) were obtained by Yu. P. Krasovskii by methods of the theory of cones (^{5}).

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