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Abstract

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LOGARITHMIC CRITERION FOR SUPERCONDUCTIVITY

(Presented by Academician N. N. Bogolyubov on 8 III 1961)

N. N. Bogolyubov and the author ([1]), on the basis of a detailed analysis of the effects of the Coulomb interaction on the effect of superconductivity, at one time came to the conclusion that the criterion for superconductivity proposed by Bardeen, Cooper, and Schrieffer ([2]), $\rho_{ph} - \rho_c > 0$, should be replaced by the more accurate logarithmic criterion

$$\rho_{ph} - \rho_c \left(1 - \rho_c \ln \frac{\omega_{ph}}{\omega_c}\right)^{-1} > 0,$$

where the dimensionless parameters ρ_{ph} and ρ_c characterize the interaction of electrons through phonon exchange and the Coulomb interaction of electrons with one another; ω_{ph} and ω_c are connected with the energy cutoff of the indicated interactions far from the Fermi surface.

The indirect interaction due to phonon exchange is cut off in a natural way at the characteristic Debye phonon energy. The direct Coulomb interaction by itself has no such characteristic cutoff. However, in view of the presence of damping of one-electron excitations far from the Fermi surface, there arises a rather sharp limitation of the action of the Coulomb interaction to the region near the Fermi surface, where the damping is still sufficiently small*.

In work ([1]) the effects of damping were not properly taken into account. Below are set forth the results of an investigation including an explicit account of damping effects. The investigation is based on the new formulation of the Feynman diagram technique proposed by N. N. Bogolyubov ([3]). The only approximation used here is the assumption of the smallness of the superconductivity effect.

Following Bogolyubov ([3]), let us consider the Hamiltonian of the electron-phonon system with Coulomb interaction, modified by the addition of terms that create and annihilate "superconducting" pairs of electrons. The system is characterized by the chemical potential μ and is enclosed in a volume V.

$$\Omega = \Omega_0 + \Omega_{\text{int}} + \Omega_\lambda, \tag{1}$$

$$\Omega_0 = \sum_{k,\sigma}(E(k)-\mu)a^+_{k,\sigma}a_{k,\sigma} + \sum_q \omega(q)b^+_q b_q, \tag{2}$$

$$\Omega_{\text{int}} = \sum_q g(q)\sqrt{\frac{\omega(q)}{2V}}\sum_{k,\sigma} a^+_{k+q,\sigma}a_{k,\sigma}(b_q + b^+_{-q})+$$

$$+\frac{1}{2V}\sum_{q\neq 0} v(q)\sum_{k_1,\sigma_1,k_2,\sigma_2} a^+_{k_1+q,\sigma_1}a^+_{k_2-q,\sigma_2}a_{k_2,\sigma_2}a_{k_1,\sigma_1}, \tag{3}$$

$$\Omega_\lambda = \sum_k \lambda(k)(a_{k,+}a_{-k,-} + a^+_{-k,-}a^+_{k,+}).$$

The creation and annihilation operators $a^+_{k,\sigma}$ and $a_{k,\sigma}$ of an electron in a state with wave vector k and spin σ satisfy Fermi

* This circumstance was noted by J. Bardeen in his lecture at the Institute for Physical Problems at P. L. Kapitsa' s seminar on 14 IX 1960.

anticommutation relations. The creation and annihilation operators b^+_q and b_q of a phonon in a state with wave vector q satisfy Bose commutation relations. In the sums over q the terms with $q = 0$ should be excluded, which is connected with taking into account the cloud of compensating positive charge.

Let us introduce, for Ω, the following electron Green functions*:

$$F(k, t'-t) = \langle T(a_{k,+}(t')a^+_{k,+}(t))\rangle; \qquad \Phi(k, t'-t) = \langle T(a^+_{k,+}(t')a^+_{-k,-}(t))\rangle, \tag{4}$$

where the averaging and the transition to time-dependent electron operators are carried out with the aid of the full operator Ω. In the diagram representation $F(k, t'-t)$ corresponds to the sum of all possible diagrams with one incoming and one outgoing external line, while $\Phi(k, t'-t)$ corresponds to the sum of all possible diagrams with two incoming (or with two outgoing) external lines. The Green functions $F(k, t)$ and $\Phi(k, t)$ satisfy the relations $F(k, t) = F(-k, t)$; $\Phi(k, t) = \Phi(-k, t)$; $\Phi(k, t) = \Phi(k, -t)$.

In the absence of interaction we have, in the energy representation,

$$F_0(k, E) = \frac{\hbar}{i}\frac{1}{2}\left\{\frac{1}{-E+\varepsilon(k)-i\varepsilon}\left(1+\frac{E(k)-\mu}{\varepsilon(k)}\right) - \frac{1}{E+\varepsilon(k)-i\varepsilon}\left(1-\frac{E(k)-\mu}{\varepsilon(k)}\right)\right\}, \tag{5}$$

$$\Phi_0(k,E) = \frac{\hbar}{i}\,\frac{\lambda(k)}{2\varepsilon(k)}\left\{\frac{1}{-E+\varepsilon(k)-i\varepsilon} + \frac{1}{E+\varepsilon(k)-i\varepsilon}\right\}, \tag{6}$$

where the notation $\varepsilon^2(k) = (E(k)-\mu)^2 + \lambda^2(k)$ has been introduced.

Let us introduce, along with the Green functions $F(k,E)$ and $\Phi(k,E)$, the corresponding mass operators $X(k,E)$ and $Y(k,E)$, which are related to them by the relations

$$\Phi(k,E)Y(k,E) - F(k,E)X(k,E) = \hbar/i;$$

$$F(k,E)Y(k,E) + \Phi(k,E)X(k,-E) = 0. \tag{7}$$

In the diagram representation the mass operators $X(k,E)$ and $Y(k,E)$ are represented by sums of all diagrams, respectively with incoming and outgoing external lines or with two incoming (or outgoing) external lines, which cannot be split into two parts connected only by one electron line.

Let us now construct an asymptotic equation of superconductivity for $Y(k,E)$, suitable for studying the effect of superconductivity when the latter is small. The starting point will be the following approximate relation, the idea for composing which belongs to N. N. Bogolyubov:

$$Y(k,E) = -\frac{1}{2\pi i}\,\frac{1}{V}\sum_{k'}\int_{-\infty}^{+\infty} dE'\, J(k,E;k',E')\,\frac{Y(k',E')}{(\hbar/i)^2 G^{-1}(k',E')G^{-1}(k',-E') + Y^2(k',E')}. \tag{8}$$

In the diagrams for $Y(k,E)$ we retain only the principal dependence on Φ or Y. The quantities $J(k,E;k',E')$ and $G(k',E')$ are taken from the ordinary diagram technique, in which $\Phi = 0$. $J(k,E;k',E')$ represents the sum of all possible diagrams with two incoming and two outgoing ends which cannot be split into two parts connected with each other only by two electron lines. $G(k',E')$ represents the sum of ordinary self-energy diagrams with one incoming and one outgoing electron line.

* Such Green functions were first introduced by S. T. Belyaev ([4]) in connection with problems of superfluidity and by L. P. Gor' kov ([5]) in connection with problems of superconductivity.

Near the Fermi surface

$$G(k,E) \simeq \frac{\hbar}{i}\,\frac{Z(k)}{-E+E(k)-\mu-i\Gamma(k)} \quad \text{as } |k| \to k_F. \tag{9}$$

The limiting wave vector is determined from the condition $E(k_F) - \mu = 0$. The damping $\Gamma(k)$ is negative for $|k| < k_F$, positive for $|k| > k_F$, and vanishes for $|k| = k_F$; moreover, $|\Gamma(k)| \ll |E(k) - \mu|$ as $|k| \to k_F$.

Using (9) and finding the first two terms of the asymptotic expansion of the right-hand side of (8) for small Y (the terms $Y \ln Y$ and Y), after simple but rather cumbersome manipulations we obtain

$$\begin{aligned} Y(k,E) = \frac{1}{2\pi^2}\,\frac{k_F^2 Z^2(k_F)}{E'(k_F)}\,Y(k_F,0)\,K(k,E;k_F,0)\,\ln\frac{Y(k_F,0)}{2\omega} \\ - \frac{i}{4\pi^3}\int_0^{+\infty} dk'\,\ln\frac{E'(k_F)|k'-k_F|}{\omega Z(k_F)}\,\frac{d}{dk'}\left({k'}^2(k'-k_F)\right)\int_{-\infty}^{+\infty} dE'\,K(k,E;k',E') \\ \times\left(\frac{i}{\hbar}\right)^2 G(k',E')G(k',-E')Y(k',E'), \end{aligned} \tag{10}$$

where $K(k,E;k',E')$ is the angular average of $J(k,E;k',E')$, and ω has been introduced on dimensional grounds.

The asymptotic superconductivity equation (10) is suitable for a complete study of the superconductivity effect when the latter is small. It automatically includes a full account of the effects of the Coulomb interaction, retardation effects in the electron–phonon interaction, and damping effects.

Let us formulate a criterion for superconductivity on the basis of the asymptotic superconductivity equation (10). We make the substitution of the unknown function

$$f(k,E) = \frac{Y(k,E)}{Y(k_F,0)\ln\frac{Y(k_F,0)}{2\omega}}, \qquad Y(k,E) = 2\omega\,\frac{f(k,E)}{f(k_F,0)}\,e^{1/f(k_F,0)}. \tag{11}$$

Then an interaction form close to critical, for which the superconductivity effect occurs, satisfies the condition $f(k_F,0) < 0$; an interaction form close to critical, for which the superconductivity effect does not occur, satisfies the condition $f(k_F,0) > 0$. The critical interaction form itself can be found from the relation $f(k_F,0) = 0$. The equation for $f(k,E)$ has the form

$$\begin{aligned} f(k,E) = \frac{1}{2\pi^2}\,\frac{k_F^2 Z^2(k_F)}{E'(k_F)}\,K(k,E;k_F,0) \\ - \frac{i}{4\pi^3}\int_0^{+\infty} dk'\,\ln\frac{E'(k_F)|k'-k_F|}{\omega Z(k_F)}\,\frac{d}{dk'}\left({k'}^2(k'-k_F)\right)\int_{-\infty}^{+\infty} dE'\,K(k,E;k',E') \\ \times f(k',E')\left(\frac{i}{\hbar}\right)^2 G(k',E')G(k',-E'). \end{aligned} \tag{12}$$

Equation (12), unlike (10), is a linear integral equation.

Let us give the result of solving (12) in the case of a weak interaction,

$$Y(k,E) = 2\omega \, \frac{K(k,E;k_F,0)}{K(k_F,0;k_F,0)} \, e^{-1/\rho}, \tag{13}$$

where

$$\rho = -\frac{1}{2\pi^2} \, \frac{k_F^2 Z^2(k_F)}{E'(k_F)} \, K(k_F,0;k_F,0), \tag{14}$$

$$\ln\omega = \frac{i}{2\pi} \int_0^{+\infty} dk' \ln \frac{E'(k_F)|k'-k_F|}{Z(k_F)} \, \frac{d}{dk'} \left(\frac{{k'}^2}{k_F^2}(k'-k_F) \times \right.$$
$$\times \int_{-\infty}^{+\infty} dE' \, \frac{K(k',E';k_F,0)}{K(k_F,0;k_F,0)} \, \frac{K(k_F,0;k',E')}{K(k_F,0;k_F,0)} \left(\frac{i}{\hbar}\right)^2 G(k',E')G(k', \tag{15}$$

Relation (15) indicates an important additional source, besides the cutoff of $K(k,E;k',E')$, leading to the main contribution to $\ln\omega$, namely, the cutoff of $G(k',E')$, when damping begins to be effective.

Let us now turn to the real case of weak interaction. Consider a model interaction for which (12) can be solved without any additional approximations. Let $Z(k) = Z(k_F) = 1$,

$$K(k,E;k',E') = K(k,k') e^{-\frac{i}{\hbar}(E-E')\varepsilon}, \tag{16}$$

where ε is infinitesimal and

$$K(k,k') = -2\pi^2 \frac{E'(k_F)}{{k'}^2} \rho_{ph} \theta_1(k')\theta_1(k) + 2\pi^2 \frac{E'(k_F)}{{k'}^2} \rho_c \theta_2(k)\theta_2(k'). \tag{17}$$

We neglect retardation of the interaction and, moreover, represent the interaction in the form of the sum of electron-phonon and Coulomb interactions, uniformly concentrated in two different energy layers near the Fermi surface: $\theta_1(k) = 1$ for $|k-k_F| < \Delta_1$, $\theta_2(k) = 1$ for $|k-k_F| < \Delta_2$, otherwise $\theta_1(k) = 0$ and $\theta_2(k) = 0$; $E'(k_F)\Delta_1 = \omega_{ph}$ and $E'(k_F)\Delta_2 = \omega_c$, $\omega_c > \omega_{ph}$. According to (15), the effective cutoff of the Coulomb interaction is determined by the damping of one-electron excitations in $G(k,E)$; the real cutoff of the electron-phonon interaction, also according to (15), is determined by the Debye phonon energy. We shall assume further that for $G(k,E)$ one may use expression (9), with $E(k') - \mu = E'(k_F)(k' - k_F)$, and also $Z(k_F) = 1$. Then from (12) it is immediately seen that

$$Y(k, E) = C(k) = C_1\theta_1(k) + C_2\theta_2(k), \tag{18}$$

$$C_1 + C_2 = 2\omega_{ph} e^{-1/\rho}; \qquad C_2 = \frac{2\omega_{ph}}{1 - \dfrac{\rho_{ph}}{\rho_c}\left(1 + \rho_c \ln \dfrac{\omega_c}{\omega_{ph}}\right)} e^{-1/\rho}; \tag{19}$$

$$\rho = \rho_{ph} - \rho_c \left(1 - \rho_c \ln \frac{\omega_{ph}}{\omega_c}\right)^{-1}. \tag{20}$$

From (20) there follows directly the logarithmic criterion of superconductivity, in which ω_c is determined by the damping of one-electron excitations far from the Fermi surface.

Let us note that, by virtue of (19), only ω_{ph}, and not any combination of ω_{ph} and ω_c, enters the preexponential factor in the energy gap; this leads to the correct explanation of the isotope effect. This circumstance is by no means trivial, since we do not assume the Coulomb interaction to be small.

In conclusion the author considers it his pleasant duty to express gratitude to N. N. Bogolyubov for a number of valuable discussions.

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