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Abstract

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SMOOTHNESS PROPERTIES OF GENERALIZED POTENTIALS OF AN ELLIPTIC OPERATOR

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In the present note the differential properties of generalized potentials of a second-order elliptic operator are investigated. Analogous results for potentials of the Laplace operator were obtained by Kh. L. Smolitskii (^{1}).

1°. **A special problem.** Here we shall consider a certain auxiliary problem, which, however, is also of independent interest, for example, in a number of questions of mechanics. Denote by g_1 an arbitrary open N-dimensional domain lying, together with its boundary C, inside the domain g, bounded by the surface Γ, and by g_2 the domain $g \setminus (g_1 + C)$. Consider the problem:

$$\begin{gathered} Lu = 0 \quad \text{in the domain } g_1, \\ Lu = 0 \quad \text{in the domain } g_2, \\ [u]\big|_C = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\Big|_C = \theta(x), \qquad u\big|_\Gamma = 0. \end{gathered} \tag{1}$$

Here L is a linear elliptic operator of second order

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u, \tag{2}$$

defined in the domain $(g + \Gamma)$; $c(x) \le 0$; $[u]\big|_C = u\big|_{C-0} - u\big|_{C+0}$;

$$\left[\frac{\partial u}{\partial \nu}\right]\Big|_C = \frac{\partial u}{\partial \nu}\Big|_{C-0} - \frac{\partial u}{\partial \nu}\Big|_{C+0};$$

$\partial/\partial\nu$ is differentiation along the conormal exterior with respect to the domain g_1; the symbols $C-0$ and $C+0$ denote that limiting values are taken, respectively, from the inner and outer sides of the surface C (with respect to g_1).

By a solution of problem (1) from the class* $C^{(n,\mu)}$ $(n \ge 2)$ we shall mean a function which: 1) belongs to the class $C^{(0)}$ in the domain $(g + \Gamma)$, to the class

$C^{(n,\mu)}$ in each of the closed domains $(g_1 + C)$ and $(g_2 + C + \Gamma)$; 2) satisfies all the conditions of problem (1) in the usual classical sense.

Theorem 1. *If the coefficients of the operator L belong to the class $C^{(n-2,\mu)}$ in the closed domain $(g + \Gamma)$, while $\theta(x)$ belongs to the class $C^{(n-1,\mu)}$ on the surface C, and the boundary surfaces C and Γ belong to the class** $A^{(n,\mu)}$, then for every solution of problem (1) from the class $C^{(n,\mu)}$ there holds*

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* For the definition of the classes $C^{(n)}$ and $C^{(n,\mu)}$, see $(^2)$, p. 10.
** For the definition of the class $A^{(n,\mu)}$, see $(^2)$, pp. 10-11.

the following a priori estimate is valid*

$$u_{n,\mu} = O(\theta_{n-1,\mu} + \theta_0) \quad (n \geqslant 2). \tag{3}$$

The constant bounding the growth of the O-terms depends on $A_{n-2,\mu}$, $B_{n-2,\mu}$, $C_{n-2,\mu}$, A_0, B_0, C_0, and also on the form of the domains g_1 and g_2.

For problem (1), the uniqueness theorem is proved without difficulty (see, for example, $(^3)$). The following theorem (of existence) is also valid:

Theorem 2. *Under the conditions of Theorem 1 there exists (and moreover a unique) solution of problem (1) of class $C^{(n,\mu)}$ $(n \geqslant 2)$.*

The proof of Theorem 2 is carried out by the method of continuation with respect to a parameter on the basis of estimate (3) and using the fact that, for the particular case of the operator L, the existence of a solution of problem (1) of class $C^{(n,\mu)}$ has already been proved (see $(^4)$).

2°. Estimates of generalized potentials. The unique solution of problem (1) is representable in the form of a generalized simple-layer potential

$$u(x) = \int_C F(x, y)\theta(y)\, dy, \tag{4}$$

where $F(x, y)$ is the Green' s function of the Dirichlet problem for the operator L in the closed domain $(g + \Gamma)$. Thus, under the conditions of Theorem 1, the **generalized simple-layer potential** belongs to the class $C^{(n,\mu)}$ and is estimated by formula (3) in each of the closed domains $(g_2 + C)$ and $(g_2 + C + \Gamma)$.

Let us now consider the following boundary-value problem:

$$\begin{gathered} Lu = 0 \quad \text{in the domain } g_1, \\ Lu = 0 \quad \text{in the domain } g_2, \\ [u]\big|_C = \varphi, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_C = 0, \qquad u\big|_\Gamma = 0. \end{gathered} \tag{5}$$

We represent the solution $u(x)$ of problem (5) in the form $u(x) = v(x) + w(x)$, where $v(x)$ is the solution of the problem

$$\begin{gathered} Lv = 0 \quad \text{in the domain } g_1, \\ v\big|_{C-0} = \varphi, \quad v \equiv 0 \quad \text{in the domain } g_2; \end{gathered} \tag{6}$$

$w(x)$ is the solution of the problem

$$\begin{gathered} Lw = 0 \quad \text{in the domain } g_1, \\ Lw = 0 \quad \text{in the domain } g_2, \\ [w]\big|_C = 0, \qquad \left[\frac{\partial w}{\partial \nu}\right]\bigg|_C = -\frac{\partial v}{\partial \nu}\bigg|_{C-0}, \qquad w\big|_\Gamma = 0. \end{gathered} \tag{7}$$

Using, for the estimate of $v(x)$, the known results of Schauder and Caccioppoli (see (^{2}), p. 144) and using formula (3) to estimate $w(x)$, we arrive at the theorem:

Theorem 3. *If the coefficients of the operator L belong to the class $C^{(n-2,\mu)}$ in the closed domain $(g+\Gamma)$; $\varphi(x)$ belongs to $C^{(n,\mu)}$ on the surface C; the boundary surfaces C and Γ belong to $A^{(n,\mu)}$, then for every solution of problem (5) of class $C^{(n,\mu)}$ the a priori estimate*

$$u_{n,\mu} = O(\varphi_{n,\mu} + \varphi_0) \quad (n \geqslant 2) \tag{8}$$

is valid.

With regard to the constant entering into this estimate, one may repeat what was said above about formula (3). From Theorem 3, as above, it follows:

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* We shall denote the sum of the maxima of the moduli of the derivatives of order n of a function $p(x)$ by p_n, and the sum of the Hölder coefficients of these derivatives, taken for the exponent μ, by $p_{n,\mu}$.

Theorem 4. Under the hypotheses of Theorem 3 there exists a (moreover unique) solution of problem (5) of the class $C^{(n,\mu)}$ $(n \geqslant 2)$.

Representing this solution further in the form of a generalized double-layer potential*

$$u(x) = \int_C Q_y F(x,y)\,\varphi(y)\,dy, \tag{9}$$

where $F(x,y)$ is the Green' s function of the Neumann problem for the operator L in the closed domain $(g+\Gamma)$, we arrive at the assertion that, under the hypotheses of Theorem 3, the generalized double-layer potential belongs to the class $C^{(n,\mu)}$ and is estimated by formula (8) in each of the closed domains (g_1+C) and $(g_2+C+\Gamma)$.

Finally, let us consider the following boundary-value problem:

$$\begin{aligned} &Lu = f && \text{in the domain } g_1, \\ &Lu = 0 && \text{in the domain } g_2, \\ &[u]\big|_C = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\Bigg|_C = 0, \qquad u\big|_\Gamma = 0. \end{aligned} \tag{10}$$

We shall represent its solution $u(x)$ in the form $u(x) = v(x) + w(x)$, where $v(x)$ is the solution of the problem

$$\begin{aligned} Lv = f & \quad \text{in the domain } g_1, \\ v\big|_{C-0} = 0, \quad v \equiv 0 & \quad \text{in the domain } g_2; \end{aligned} \tag{11}$$

and $w(x)$ is the solution of the problem

$$\begin{aligned} &Lw = 0 && \text{in the domain } g_1, \\ &Lw = 0 && \text{in the domain } g_2, \\ &[w]\big|_C = 0, \qquad \left[\frac{\partial w}{\partial \nu}\right]\Bigg|_C = -\frac{\partial v}{\partial \nu}\Bigg|_{C-0}, \qquad w\big|_\Gamma = 0. \end{aligned} \tag{12}$$

In complete analogy with the preceding, from (11) and (12) we conclude that the following holds:

Theorem 5. If the coefficients of the operator L belong to the class $C^{(n-2,\mu)}$ in the closed domain $(g+\Gamma)$; $f(x)$ belongs to $C^{(n-2,\mu)}$ in (g_1+C); the boundary surfaces C and Γ belong to $A^{(n,\mu)}$, then there exists a (moreover unique) solution of problem (10) of the class $C^{(n,\mu)}$, and for it the estimate

$$u_{n,\mu} = O\left(f_{n-2,\mu} + f_0\right) \qquad (n \geqslant 2). \tag{13}$$

is valid.

On the other hand, this solution of problem (10) can be represented in the form of a generalized volume potential:

$$u(x) = \int_{g_1} F(x;y)\, f(y)\, dy, \tag{14}$$

where $F(x,y)$ is the Green' s function of the Dirichlet problem for the operator L in the domain $(g+\Gamma)$. Thus, if the hypotheses of Theorem 5 are fulfilled, the generalized volume potential belongs to the class $C^{(n,\mu)}$ in each of the closed domains $(g_1 + C)$ and $(g_2 + C + \Gamma)$, and estimate (13) is valid for it in these domains.

Remark. It is obvious that if in formulas (4), (9), (14), by $F(x, y)$ we understand any principal fundamental solution of the equation $Lu = 0$, then one may assert that the potentials defined by these formulas will belong to the class $C^{(n,\mu)}$, and for them there will be

* For the definition of the operator Q_y, see (^{2}), p. 21.

the estimates (3), (8), and (13) are valid in each of the closed domains $(g_1 + C)$ and $(T + C + \Gamma_T)$, where T is an arbitrary domain such that $g_1 \subset T \subset g$; Γ_T is the boundary of T, if the conditions of Theorems 1, 3, and 5 are satisfied with the requirement $\Gamma \subset A^{(n,\mu)}$ replaced by the requirement $\Gamma_T \subset A^{(n,\mu)}$.

3°. **Proof of Theorem 1.** We shall outline the proof of Theorem 1, the main theorem in this paper. We use Schauder' s method, developed for obtaining a priori estimates. Fix an arbitrary point x_0 of the domain g_1 and rewrite problem 1° in the form:

$$\sum_{i,j=1}^{N} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^{N}[a_{ij}(x_0) - a_{ij}(x)]\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} - cu \equiv f_1 \quad \text{in } g_1,$$

$$\sum_{i,j=1}^{N} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^{N}[a_{ij}(x_0) - a_{ij}(x)]\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} - cu \equiv f_2 \quad \text{in } g_2, \tag{15}$$

$$[u]\big|_C = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_C = \theta(x), \qquad u\big|_\Gamma = 0.$$

By means of a change of coordinates, in the left-hand side of equations (16) we pass from operators with constant coefficients to the Laplace operator; we obtain

$$\Delta u = \bar{f}_1 \quad \text{in the domain } \bar{g}_1,$$

$$\Delta u = \bar{f}_2 \quad \text{in the domain } \bar{g}_2, \tag{16}$$

$$[u]\big|_{\bar{C}} = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_{\bar{C}} = \bar{\theta}, \qquad u\big|_{\bar{\Gamma}} = 0.$$

We represent the solution of problem (16) in the form $u(x) = v(x) + w(x)$, where $w(x)$ is the solution of a problem of the type (16) in which zero stands in place of $\bar{\theta}(x)$, and

$$v(x) = \int_{\bar{C}} K(x,y)\,\mu(y)\,dy;$$

$K(x,y)$ is the Green' s function of the Dirichlet problem for the Laplace operator in the domain $(\bar{g} + \bar{\Gamma})$, and $\mu(y)$ is chosen from the condition

$$\left[\frac{\partial v}{\partial \nu}\right]\Big|_{\bar{C}} = \bar{\theta}.$$

The function $v(x)$ is easily estimated (see (1)); and for the estimate of $w(x)$ one first considers the problem:

$$\Delta\tau = F_1 \quad \text{in the domain } \bar{g}_1 \cap \Omega(\bar{x}_0, 2\rho),$$

$$\Delta\tau = F_2 \quad \text{in the domain } \bar{g}_2, \tag{17}$$

$$[\tau]\big|_{\bar{C}} = 0, \qquad \left[\frac{\partial \tau}{\partial \nu}\right]\Big|_{\bar{C}} = 0, \qquad \tau\big|_{\bar{\Gamma}} = 0;$$

$\Omega(\bar{x}_0, 2\rho)$ is the ball of radius 2ρ with center at the point $\bar{x}_0$. Having obtained an estimate of $\tau_{n,\mu}$ in $\Omega(\bar{x}_0, \rho)$, we estimate $w(x)$ by a method close to that set forth in (^{2}).

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Note: Figure translations are in progress. See original paper for figures.