GEOPHYSICS

F. I. FRANKL and L. N. GUTMAN

1961

SovietRxiv

Abstract

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GEOPHYSICS

F. I. FRANKL and L. N. GUTMAN

A STATIONARY PROBLEM ON THE MOTION OF A COLD LAYER OF AIR OVER RUGGED TERRAIN

(Presented by Academician A. A. Dorodnitsyn, 15 VI 1961)

Generalizing the works ([1–3]), we consider a spatial stationary problem on the motion of a cold air mass over complex relief, whose characteristic horizontal scales are 10^2-10^3 km. We shall neglect the influence of turbulence, as well as the heat input from condensation of water vapor, regarding the process as adiabatic. Suppose that the motion of the warmer air mass displaced upward by the cold air does not depend on the orography and is determined by a prescribed pressure field. Let us assume that the relief, at least on one side, passes into a horizontal plain, from which the cold air arrives. The temperature of the warm air, as well as that of the undisturbed cold air, will be considered to depend only on the height z, and we prescribe it in the form $\theta_H - \gamma_T(z-H)$ in the warm air and $\theta_H - \Delta\theta - \gamma(z-H)$ in the cold air. Here $H = \text{const}$ is the thickness of the cold air mass over the plain; $\gamma_T = \text{const}$, $\gamma = \text{const}$ are the vertical temperature gradients; $\theta_H = \text{const}$ is the temperature of the warm air at $z = H$; $\Delta\theta = \text{const}$ is the temperature difference between warm and cold air at $z = H$ over the plain. $\Delta\theta$ characterizes the initial thermal difference of the air masses. The quantities H, γ_T, γ, θ_H, and $\Delta\theta$ are assumed known.

We pass to the mathematical formulation of the problem. In view of the assumptions made, the motion of the warm air mass may be considered quasi-geostrophic, and for its description one may use the usual geostrophic relations, transformed with allowance for the polytropic character of the atmosphere:

$$U(x,y,z) = U_H\left[1 - \frac{\gamma_T(z-H)}{\theta_H}\right]\left(U_H = -\frac{1}{l\rho_H}\frac{\partial P(x,y,H)}{\partial y}\right),$$

$$V(x,y,z) = V_H\left[1 - \frac{\gamma_T(z-H)}{\theta_H}\right]\left(V_H = \frac{1}{l\rho_H}\frac{\partial P(x,y,H)}{\partial x}\right). \qquad (1)$$

Here l is the Coriolis parameter; ρ_H is the mean value of the air density at height H; (x,y,z) is a rectangular right-handed (for the northern hemisphere of the Earth) coordinate system; U, V are the components of the quasi-geostrophic wind along the axes x, y; $U_H(x,y) = U(x,y,H)$; $V_H(x,y) = V(x,y,H)$; the field of the quasi-geostrophic wind, as well as the pressure field $P(x, y, z)$, will be considered known.*

According to (1), the vertical gradient of the geostrophic wind is negligibly small. Indeed, taking, for example, $\gamma_T = 7^\circ$/km, $\theta_H = 250^\circ$, $U_H = 10$ m/sec, we find that even over a vertical distance equal to 3 km the difference in the quasi-geostrophic wind will not exceed 1 m/sec. This value lies within the limits of measurement accuracy. Therefore, without introducing appreciable

* In the subsequent arguments we imagine the upper quasi-geostrophic layer analytically continued into the region actually occupied by the cold layer.

errors, we may, for not very great heights, regard the quasi-geostrophic wind as independent of height and assume, instead of (1),

$$U(x, y, z) = U_H(x, y); \qquad V(x, y, z) = V_H(x, y).$$

Let us note that, under this formulation of the problem, the general system of equations for the motion of cold air can be substantially simplified.

First of all, without introducing any additional error, we may assume that the horizontal components of the velocity of the cold air, u, v, depend only on the horizontal coordinates. Indeed, the derivatives $\partial u/\partial z$ and $\partial v/\partial z$ must be of the same order as $\partial U/\partial z$ and $\partial V/\partial z$ (since there are no reasons that could cause an essential difference between them), i.e., they are negligibly small*. This circumstance, as well as the obvious condition of pressure continuity on the interface between the warm and cold air masses $P(x, y, \eta) = p(x, y, \eta)$ ($z = \eta(x, y)$ is the equation of the interface, to be determined), suggests writing the equations of motion of the cold air at $z = \eta(x, y)$. In these equations it is expedient to express the horizontal pressure gradients through derivatives of the pressure taken along the surface $z = \eta(x, y)$. Carrying out the transformations with allowance for the equations of quasistatics, Clapeyron, and adiabaticity (Poisson), then expanding the resulting expression in a series in powers of small parameters and discarding terms of higher order of smallness, we shall have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\mu\frac{\partial \zeta}{\partial x} - \nu\frac{\partial \zeta^2}{\partial x} + l(v - V_H); \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\mu\frac{\partial \zeta}{\partial y} - \nu\frac{\partial \zeta^2}{\partial y} - l(u - v_H), \tag{3}$$

where $\zeta = \eta - H$; $\mu = g\,\Delta\theta/\theta_H$; $\nu = g(\gamma_a - \gamma)/2\theta_H$, g is the acceleration of gravity; γ_a is the dry-adiabatic gradient. In these equations, describing the motion of the cold layer, all functions depend only on two variables x and y. Let us emphasize the physical meaning of the first two terms on the right-hand side of (2), (3). If $\mu\,\partial\zeta/\partial x$ and $\mu\,\partial\zeta/\partial y$ represent the component of the pressure gradient caused by changes in the thickness of the flow, then $\nu\,\partial\zeta^2/\partial x$ and $\nu\,\partial\zeta^2/\partial y$ are the component caused by a change in the temperature of the flow.

Let us now turn to the equation of continuity. With high accuracy it may be written in the form (see (4))

$$\frac{\partial u e^{-\sigma z}}{\partial x} + \frac{\partial v e^{-\sigma z}}{\partial y} + \frac{\partial w e^{-\sigma z}}{\partial z} = 0,$$

where $\sigma = \partial \ln \rho/\partial z \approx$ const. Integrating this equation with respect to z from δ to η ($z = \delta(x, y)$ is the equation of the terrain), taking into account the flow conditions

$$w_{z=\delta} = u\frac{\partial \delta}{\partial x} + v\frac{\partial \delta}{\partial y}, \qquad w_{z=\eta} = u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y},$$

then expanding the exponentials in series in powers of $\sigma\eta$ and $\sigma\delta$ and retaining only the principal terms of the series, we arrive at the following final form of the continuity equation:

$$\frac{\partial u h(1-\sigma\delta)}{\partial x} + \frac{\partial v h(1-\sigma\delta)}{\partial y} = 0, \tag{4}$$

where $h = h(x, y) \approx \eta - \delta$ is the thickness of the cold layer, which we assume does not exceed 2–3 km. It should be borne in mind that the term $\sigma\delta$, which takes into account the decrease of the static air density with height, will be small, and it may be discarded so long as $\delta \ll 2$ km.

* The same conclusion is reached by the results of work (3), in which an analogous plane problem was considered with allowance for $\partial u/\partial z$ (see the figure in (3)).

We shall show that the system of equations (2), (3) admits the existence of first integrals. For this purpose, introducing the stream function ψ from the relations

$$u = \frac{1}{h(1-\delta\zeta)}\frac{\partial \psi}{\partial y}; \qquad v = -\frac{1}{h(1-\delta\zeta)}\frac{\partial \psi}{\partial x}, \tag{5}$$

we multiply (2) by u, (3) by v, and after adding these equations arrive at the energy integral, which generalizes the well-known Bernoulli integral:

$$\frac{u^2+v^2}{2} + \mu\zeta + \nu\zeta^2 + \frac{P(x, y, H)}{\rho H} = f(\psi), \tag{6}$$

where $f(\psi)$ is a function whose form is determined from the boundary conditions of the problem. Now subtracting, term by term, the result of differentiating (6) with respect to y from equation (3), we obtain the equation for the vortex

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = l - (1 - \delta\zeta) h \frac{df}{d\psi}. \tag{7}$$

Thus, the problem has been reduced to the solution of two equations (6) and (7), while u and v are expressed in terms of ψ by means of equalities (5).

The corresponding boundary conditions will be: $u = U_H$; $v = V_H$; $\zeta = 0$ over the plain. They take into account the absence of perturbations of the meteorological elements in the incident flow. We note that $U_H(x, y)$, $V_H(x, y)$ differ somewhat from the wind which should occur in the cold air over the plain. However, owing to the smallness of the perturbations of the air temperature in comparison with its mean value, these differences are negligible.

With the aim of drawing some general conclusions, let us substitute u and v from (5) into (7) and, after rather cumbersome transformations, arrive at the equation:

$$(a^2 - u^2) \frac{\partial^2 \psi}{\partial x^2} - 2uv \frac{\partial^2 \psi}{\partial x \partial y} + (a^2 - v^2) \frac{\partial^2 \psi}{\partial y^2} + ... = 0, \tag{8}$$

where $a = \sqrt{(\mu + 2\nu\zeta)h}$ is the generalized velocity of surface waves. In (8), the unwritten terms do not contain second derivatives of the stream function. Thus, equation (8) will be of hyperbolic type for $a^2 > u^2 + v^2$ and elliptic for $a^2 < u^2 + v^2$. It is not difficult to calculate that, for those values of μ, ν, h and ζ which occur in reality, the speed a will be of the same order as the wind speeds observed in nature, i.e., of the order of 10 m/sec. Consequently, the speed of surface waves in problems of flow around mountains plays the same role as the speed of sound for plane parallel stationary flows of gases. Hence we come to the conclusion that, when air flows over a mountain ridge, when it flows through the narrow throat of a gorge, when it flows around a cape, etc., a transition from subcritical velocities to supercritical ones is possible. In particular, there is reason to assert that it is precisely in this way that one may interpret the phenomenon of the bora studied by the authors ([3]). In some cases one should expect the realization in the atmosphere of hydrodynamic effects of the type of a pressure jump or a hydraulic jump*.

In conclusion we express our gratitude to Corresponding Member of the Academy of Sciences of the USSR I. A. Kibel for valuable advice.

High-Mountain Geophysical Institute
Academy of Sciences of the USSR

Received
14 VI 1961

CITED LITERATURE

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* According to an oral communication by Prof. A. M. Gusev, a similar phenomenon is indeed sometimes observed in the Novorossiisk bora.

Note: Figure translations are in progress. See original paper for figures.