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Abstract

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On the Estimation of the Spectrum of a Stationary Gaussian Process

(Presented by Academician A. N. Kolmogorov, 24 VI 1961)

1. Consider a real stationary Gaussian process $\{x_n\}$ with discrete time $n = \dots, -1, 0, 1, \dots$ and $\mathbf{E}x_n = 0$. We shall denote by $F(\lambda)$ and $f(\lambda)$ the spectral function (s.f.) and the spectral density (s.d.) of the process. In the present note we set forth some results concerning estimation of the s.f. of the process $\{x_n\}$ from a sample $x_1, x_2, \dots, x_N$ of the process $\{x_n\}$ (see, for example, ([1,2])). In view of the real-valuedness of the process, in what follows we restrict ourselves to estimating the spectrum of the process on the interval $[0, \pi]$ and call the s.f. of the process the function $F(\lambda) - F(0)$, denoting it also by $F(\lambda)$.

As an estimate for $F(\lambda)$ one often uses the statistic

$$F_N^*(\lambda) = \frac{1}{2\pi N} \int_0^\lambda \left| \sum_{j=1}^N e^{i\lambda j} x_j \right|^2 d\lambda.$$

Since, for $|j| < N$,

$$\int_0^\pi e^{i\lambda j}\, dF_N^*(\lambda) = \frac{1}{2N} \sum_{k=1}^{N-|j|} x_{k+|j|} x_k,$$

it follows that for any ergodic process* $\{x_n\}$, with probability 1,

$$\int_0^\pi e^{i\lambda j}\, dF_N^*(\lambda) \underset{N\to\infty}{\longrightarrow} \int_0^\pi e^{i\lambda j}\, dF(\lambda), \qquad j = \dots, -1, 0, 1, \dots,$$

and therefore, with probability 1, $L(F_N^*, F) \underset{N\to\infty}{\longrightarrow} 0$, where in general $L(F, G)$ denotes the Lévy distance between the (unnormalized) distribution functions F, G. Unfortunately, as the following Theorem 1 shows (this theorem is an answer to a question posed to the author by A. N. Kolmogorov), the convergence to zero of the distance $L(F_N^*, F)$ is not uniform.

Theorem 1. There exists an absolute constant $\beta_0 > 0$ such that, for all sufficiently small ε,

$$\sup \mathbf{P}\{L(F_N^*, F) \geqslant \varepsilon\} \geqslant \beta_0 - \left(\frac{9}{4}\varepsilon\right)^{1/4}, \tag{1}$$

where the supremum is taken over all absolutely continuous s.f. $F(\lambda)$ such that $F(\pi) = 1$.

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* Recall that a stationary Gaussian process is ergodic if and only if the s.f. $F(\lambda)$ is continuous.

Remark 1. Theorem 1 remains true if, instead of the d.f. $F(\lambda)$, one estimates the function $F(\pi) - F(\lambda)$; the interval $[0, \pi]$ may be replaced by the interval $[\delta, \pi]$, $\delta > 0$.

Remark 2. The right-hand side of inequality (1) will not change if one takes the infimum of the left-hand side over all estimates $F_N^* = F_N^*(\lambda; x_1, \dots, x_N)$ having the following properties: 1) for fixed $x_1, \dots, x_N$, F_N^* is a nondecreasing function of λ; 2)

$$F_N^*(\pi; x_1, \dots, x_N) = \frac{x_1^2 + \dots + x_N^2}{2}.$$

In the proof of Theorem 1 we choose a sequence of absolutely continuous d.f.' s converging to a step d.f. If one restricts oneself to classes of d.f.' s that are all uniformly separated from discontinuous d.f.' s, then, as follows from Theorem 2, inequality (1) will no longer hold. Denote by $\mathfrak{F}$ some class of equicontinuously continuous d.f.' s with $F(\pi) = 1$. Let $\omega_F(\delta)$ denote the modulus of continuity of the function $F(\lambda)$. Put $\omega_{\mathfrak{F}}(\delta) = \sup_{F \in \mathfrak{F}} \omega_F(\delta)$. Clearly, $\omega_{\mathfrak{F}}(\delta) \downarrow 0$, $\delta \to 0$.

Theorem 2. Whatever the number $\varepsilon > 0$, as $N \to \infty$

$$\sup_{F \in \mathfrak{F}} \mathbf{P}\{L(F_N^*, F) \geqslant \varepsilon\} \to 0, \tag{2}$$

and the left-hand side in (2) does not exceed

$$C \, \frac{\min\limits_{\delta} \left(\dfrac{1}{N\delta} + \omega_{\mathfrak{F}}(2\delta)\right)}{\varepsilon^3},$$

where C is an absolute constant.

2. Suppose here that the d.f. $F(\lambda)$ is absolutely continuous and

$$\int_{-\pi}^{\pi} f^2(\lambda)\, d\lambda < \infty.$$

Let P_N denote the probability measure in the space $C[0, \pi]$ generated by the random process $\sqrt{N}\,[F_N^*(\lambda) - F(\lambda)]$; let P denote the probability measure in

$C[0,\pi]$ generated by the Gaussian process $\xi(\lambda)$ with $\xi(0)=0$, $\mathbf{E}\xi(\lambda)=0$,

$$\mathbf{E}\xi(\lambda)\xi(\mu) = 2\pi \int_0^{\min(\lambda,\mu)} f^2(\lambda)\, d\lambda.$$

It is not hard to show that

$$\mathbf{P}\left\{\max_{0\le\lambda\le\pi} |\xi(\lambda)| \le z\right\} = \sum_{k=-\infty}^{\infty} (-1)^k \left[\Phi\left(\frac{(2k+1)z}{\sqrt{2\pi G}}\right) - \Phi\left(\frac{(2k-1)z}{\sqrt{2\pi G}}\right)\right] = \Delta\left(\frac{z}{\sqrt{2\pi G}}\right),$$

where

$$G = \int_0^{\pi} f^2(\lambda)\, d\lambda,$$

and $\Phi(z)$ is the normal distribution function.

Theorem 3 (cf. ([1,3])). Suppose the following conditions are satisfied:

1) The d.f. $F(\lambda)$ has no intervals of constancy.
2) $$\int_{-\pi}^{\pi} (f(\lambda))^{2+\delta}\, d\lambda < \infty$$
   for some $\delta > 0$.

Then the sequence of measures P_N converges weakly, as $N\to\infty$, to the measure P. In particular,

$$\mathbf{P}\left\{\max_{0\le\lambda\le\pi} \sqrt{N}\, |F_N^*(\lambda) - F(\lambda)| \le z\right\} \underset{N\to\infty}{\longrightarrow} \Delta\left(\frac{z}{\sqrt{2\pi G}}\right).$$

For weak convergence of measures, see, for example, the work ([4]), the methods of which are substantially used in the proof of Theorem 3.

Corollary (cf. (1[3])). Under the conditions of Theorem 2 the relation

$$F_N^*(\lambda) - z\sqrt{\frac{2\pi G^*}{N}} \le F(\lambda) \le F_N^*(\lambda) + z\sqrt{\frac{2\pi G^*}{N}},$$

$$G^* = \frac{1}{4\pi N^2} \sum_{\nu=0}^{[KN^\alpha]} \left(\sum_{j=1}^{N-\nu} x_{j x_{j+\nu}}\right)^2, \qquad 0<\alpha<1, \quad K>0,$$

holds with probability tending to $\Delta(z)$ as $N\to\infty$.

Remark 1. Theorem 2 remains valid if the measures P_N are replaced by the measures $P_N(W_N)$, generated by the processes $\sqrt{N}\,[F_N^*(\lambda, W_N) - F(\lambda, W_N)]$, where

$$F_N^*(\lambda, W_N) = \frac{1}{2\pi N} \int_0^\pi \left| \sum_{j=1}^N x_j e^{ilj} \right|^2 W_N(\lambda - l)\, dl,$$

$$F(\lambda, W_N) = \int_0^\pi f(l) W_N(\lambda - l)\, dl,$$

$$W_N(\lambda) = \int_{-\pi}^\lambda w_N(\lambda)\, d\lambda, \qquad w_N(\lambda) \geq 0, \qquad W_N(\pi) = 1,$$

$$\lim W_N(\lambda) = \begin{cases} 0, & \lambda < 0, \\ 1, & \lambda > 0. \end{cases}$$

Remark 2. As the following example shows, the results of Theorem 2 cannot be substantially strengthened. Consider the Gaussian process $\{x_n\}$ with spectral density $f_a(\lambda) = \frac{1}{|\lambda|^a}$, $\frac{1}{2} < a < 1$, and denote by $P_N^{(h)}$ the measures in $C[0, \pi]$ generated by the processes $N^{1-h}[F_N^*(\lambda) - F(\lambda)]$. It turns out that for $h < a$ the sequence $P_N^{(h)}$ is not compact; for $h > a$ the sequence $P_N^{(h)}$ converges to the zero element of the space $C[0, \pi]$; for $h = a$ the measure limiting for $P_N^{(h)}$ is in any case non-Gaussian.

3. At present little is known about the quality of the estimates $F_N^*(\lambda)$. The theorem given in this section suggests that, for a sufficiently broad class of processes, the estimates $F_N^*(\lambda)$ already for small N give a good approximation to $F(\lambda)$.

Theorem 4. If $0 < m \leq f(\lambda) \leq M < \infty$, then for any $\delta > 0$

$$\mathbf{P}\left\{ \max_{\delta \leq \lambda \leq \pi} \sqrt{N}\, |F_N^*(\lambda) - F(\lambda)| > 3Mz + \sqrt{G} \right\} \leq C_1 M^{5/2} e^{-\varepsilon z^2/2},$$

where $\varepsilon < 1$,

$$1 \leq z \leq C_2 \frac{m}{M} \sqrt{N} \max\left(\sqrt{\delta - \frac{2}{N}}, \sin\delta \right),$$

and the constants C_1 and C_2 depend only on ε.

4. The theorems stated above remain true for processes with continuous time and for random fields. We shall not give the formulations of the corresponding theorems, since in essence they do not differ from their discrete analogues with one-dimensional time*. -

* For processes with continuous time, the convergence of the measures from Theorem 3 is considered not in $C[0,\infty]$, but in $C[0,a]$, $a<\infty$.

let us note only that the estimates will now have the following form:

$$F_T^*(\lambda) = \frac{1}{2\pi T}\int_0^\lambda \left|\int_0^T x_t e^{it\lambda}\,dt\right|^2 d\lambda$$

in the case of continuous time;

$$F^*_{N_1\dots N_k}(\lambda_1,\dots,\lambda_k) =$$

$$= \frac{1}{(2\pi)^k N_1\dots N_k}\int_{-\pi}^{\lambda_1}\dots\int_{-\pi}^{\lambda_k}\left|\sum_{j_1,\dots,j_k=1}^{N_1,\dots,N_k} e^{i(j_1\lambda_1+\dots+j_k\lambda_k)}x_{j_1\dots j_k}\right|^2 d\lambda_1,\dots,d\lambda_k$$

in the case of a discrete field with k-dimensional time.

5. In this paragraph we shall indicate methods of estimating the s.f. of generalized Gaussian stationary processes $X(\varphi)$, where $\varphi\in\mathcal{D}$, and by $\mathcal{D}$ is denoted the class of infinitely differentiable complex-valued functions with compact support (5–7). In what follows we use the following notation: $\tau_s\varphi(t)=\varphi(t+s)$, $\mathfrak{F}\varphi(\lambda)=\int e^{-it\lambda}\varphi(t)\,dt$. Let $\varphi=\varphi(t)$ be a fixed function from $\mathcal{D}$. Then $Y(s)=X(\tau_{-s}\varphi)$ is an ordinary stationary Gaussian process with s.f.

$$F_Y(\lambda)=\int_{-\infty}^{\lambda}|\mathfrak{F}\varphi(\lambda)|^2\,dF(\lambda)$$

and s.d. $f_Y(\lambda)=|\mathfrak{F}\varphi(\lambda)|^2 f(\lambda)$. Consequently, if $F_Y^*(\lambda)$ and $f_Y^*(\lambda)$ are any estimates for the s.f. and s.d. of the process $Y(s)$, then they are simultaneously estimates for

$$\int_{-\infty}^{\lambda}|\mathfrak{F}\varphi(\lambda)|^2\,dF(\lambda),\quad |\mathfrak{F}\varphi(\lambda)|^2 f(\lambda).$$

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REFERENCES

1. U. Grenander, M. Rosenblatt, *Statistical Analysis of Stationary Time-Series*, Stockholm, 1956.

2. U. Grenander, M. Rosenblatt, Proc. 3-d Berkeley Symp. Math. Stat. and Prob., Berkeley—Los Angelos, 1956.

3. U. Grenander, M. Rosenblatt, Ann. Math. Stat., 24, 537 (1953).

4. Yu. V. Prokhorov, Theory of Probability and Its Applications, 1, 2 (1956).

5. I. M. Gel' fand, DAN, 100, No. 5 (1955).

6. K. Itô, Mem. College Sci. Univ. Kyoto, Ser. A, 28, No. 3 (1954).

7. A. M. Yaglom, Theory of Probability and Its Applications, 2, 3 (1957).

Note: Figure translations are in progress. See original paper for figures.