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Abstract

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SOME ESTIMATES CONCERNING ALGEBRAIC HYPERSURFACES AND DERIVATIVES OF RATIONAL FUNCTIONS

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1, 0. Let $P_n(x) = P_n(x_1, x_2, \dots, x_k) \not\equiv 0$ be a real algebraic polynomial of degree n in the aggregate of variables $x_1, x_2, \dots, x_k$ $(k \geqslant 1)$; let D_k be the closed unit cube of the k-dimensional real Euclidean space E_k (the faces of D_k need not be parallel to the coordinate planes); let $\Gamma_k(P_n)$ be the hypersurface lying in the space E_k and defined by the equation $P_n(x_1, x_2, \dots, x_k) = 0$; let $\gamma_k(P_n) = \Gamma_k(P_n) \cap D_k$; and let $N_k(\varepsilon, P_n)$ be the least number of points forming, for a given $\varepsilon > 0$, an ε-net on $\gamma_k(P_n)$.

1,1. Theorem 1. *For each $k = 1, 2, \dots$ there exists a constant N_k $\left(1 \leqslant N_k \leqslant 2^{1/2(k+4)(k-1)}\right)$, depending only on k, such that*

$$N_k(\varepsilon, P_n) \leqslant N_k n(\varepsilon^{1-k} + n^{k-1})$$

for any polynomial P_n and any $\varepsilon > 0$.

The theorem can be proved by induction on k, using the fact that $\Gamma_k(P_n)$ has no more than $2nk^2$ bounded connected components ([1], [2]).

Corollary 1. *The k-dimensional volume of the ε-neighborhood of the set $\gamma_k(P_n)$ does not exceed $V_k[\varepsilon n + (\varepsilon n)^k]$, where the constant V_k depends only on k.*

1,2. We introduce new notation. Let Ω_k be the k-dimensional unit ball belonging to E_k, with center at the origin, and also the k-dimensional volume of this ball:

$$\Omega_k = \pi^{n/2}/\Gamma(n/2+1);$$

let ω_k be the surface of the ball Ω_k, and also the $(k-1)$-dimensional area of this surface:

$$\omega_k = 2\pi^{n/2}/\Gamma(n/2);$$

let e be a point of the sphere ω_k, and also the unit vector issuing from the origin and having the point e as its endpoint; let $\sigma(e)$ be the $(k-1)$-dimensional hyperplane passing through the origin and having the vector e as its normal vector; let $L(T, e)$ be the straight line passing through the point $T \in \sigma(e)$ orthogonally to $\sigma(e)$; let S be a $(k-1)$-dimensional hypersurface lying in E_k, and also the $(k-1)$-dimensional measure of this hypersurface; and let $\nu(T,e,S)$ be the number of connected components of the set $S \cap L(T,e)$ $(0 \leqslant \nu(T,e,S) \leqslant \infty)$.

Lemma. *For a piecewise smooth* $*$ $(k-1)$*-dimensional hypersurface* S*,*

$$S = \frac{1}{2\Omega_{k-1}} \int_{\omega_k} d_e \omega_k \int_{\sigma(e)} \nu(T,e,S)\, d_T \sigma(e) \quad **.$$

The proof of the lemma for the case when S is a piece of a plane is obtained easily if one observes that in this case

$$\int_{\sigma(e)} \nu(T,e,S)\, d_T \sigma(e)$$

is the $(k-1)$-dimensional area of the orthogonal projection of S onto $\sigma(e)$. Since the validity of the lemma for polyhedral surfaces already follows from this, to prove it in the general case it remains only to approximate the piecewise smooth hypersurface S by a polyhedral surface Σ and pass to the limit. This lemma is a generalization of Theorem 22 from ([3]).

* A $(k-1)$-dimensional hypersurface is called piecewise smooth if it can be subdivided into a finite number of pieces with a continuously varying $(k-1)$-dimensional tangent hyperplane.

** The subscript on the differential sign indicates the variable of integration.

Taking into account that every line not lying on $\Gamma_k(P_n)$ intersects $\Gamma_k(P_n)$ in no more than n points, and that every line intersecting some domain G of the space E_k intersects the boundary S of this domain in no fewer than two places, as a consequence of the lemma we obtain the theorem:

Theorem 2. The $(k-1)$-dimensional area of that part of the surface $\Gamma_k(P_n)$ which falls in the domain $G \subset E_k$ does not exceed $\frac{1}{2}nS$, where S is the $(k-1)$-dimensional area of the boundary of the domain G.

Corollary 1. The $(k-1)$-dimensional area of the surface $\gamma_k(P_n)$ does not exceed kn.

Corollary 2. The $(k-1)$-dimensional area of that part of the hypersurface $\Gamma_k(P_n)$ which falls in an ellipsoid with semiaxes $a_1, a_2, \dots, a_k$ does not exceed

$$\pi n^{n/2} a_1 a_2 \cdots a_k / \Gamma(n/2).$$

For even n this estimate is sharp for every set $k, a_1, a_2, \dots, a_k$.

The sharpness of the estimate is indicated by the example of the polynomial

$$P_n(x_1, x_2, \dots, x_k) = \prod_{j=1}^{n/2} \left[\sum_{i=1}^{k} \left(\frac{x_i}{b_{ji}} \right)^2 - 1 \right],$$

where b_{ji} are sufficiently close to a_i $(b_i < a_i)$.

In conclusion of this subsection we note that estimates analogous to those given above can be obtained by the same methods also for algebraic hypersurfaces (lying in E_k) having dimension less than $k-1$.

1.3. Let

$$\widetilde{P}_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

be a polynomial in the complex variable z with leading coefficient equal to 1; C is a positive constant. With the aid of the lemma given above and the theorems on transfinite diameter ([4]), the following is proved:

Theorem 3. The length of the curve $\Gamma : \ |\widetilde{P}_n(z)| = C$ does not exceed

$$4\pi n \sqrt[n]{C}.$$

The constant 4π cannot, for any n and C, be replaced by 2.

2.0. We give estimates of derivatives of rational functions.

It is well known ([5]) that the derivative of an algebraic polynomial $P(x)$ on the interval $[a, b]$ can be estimated in terms of the maximum of $|P(x)|$ on this interval, the degree of $P(x)$, and the length of $[a, b]$. From the example of the function $R_2(x) = \varepsilon/(x^2+\varepsilon)$ it is easy to see that the derivative of a rational fraction $R(x)$, in general, can no longer be estimated in a similar manner. However, if one "throws out" from the number axis, in a special way, some set e of prescribed measure $\delta > 0$, then outside this set the derivative of $R(x)$ can already be estimated in terms of the order of $R(x)$ and the maximum of its modulus ([6]). In this case the number δ enters into the estimate.

Below, with the aid of a simple method consisting in estimating the integral of a certain power of the modulus of $R'(z)$ in terms of the order of $R(z)$ and the maximum of its modulus, sharp estimates of this kind are obtained for rational functions of a real, as well as of a complex, variable. These estimates are also useful in considering polynomials $P(z)$ in cases where the known estimates for $P'(x)$ cease to be applicable (as, for example, in the case when the set E in the complex plane on which the derivative of the polynomial $P(z)$ is estimated in terms of its degree and the maximum of its modulus has a sufficiently general nature—say, is a nowhere dense set—or, in the case $E \subset (-\infty, \infty)$, does not consist of a finite number of intervals).

2.1. Let $R_n(x)$ denote a rational fraction of order not higher than n (the order of an irreducible rational fraction is the maximum of the degrees of its numerator and denominator); $g(M, R_n)$ is the set of all points $x \in (-\infty, \infty)$ for which $|R_n(x)| \le M$, where M is a positive constant.

Theorem 4. For any real rational function $R_n(x)$ and any positive number δ there exists a set

$$E_1(\delta) = E_1(\delta, M, R_n)$$

such that $\operatorname{mes} E_1(\delta) < \delta$ and, for $x \in g(M, R_n) \setminus E_1(\delta)$,

$$|R_n'(x)| \leq \frac{2}{\delta}\, nM.$$

The estimate is sharp for every set of n, M, δ. The equality sign is in fact attained only in the case of the linear function

$$R_1(x) = \pm\frac{2}{\delta}Mx + b \quad (b = \text{const}).$$

The proof is obtained from the obvious inequality

$$\int_{g(M,R_n)} |R_n'(x)|\, dx \leq 2Mn.$$

It is of interest to compare the estimate given above for $R_n'(x)$ with S. N. Bernstein’ s estimate for the derivative of a polynomial $P_n(x)$ on $[-1, 1]$:

$$|P_n'(x)| \leq (1 - x^2)^{-1/2} nM.$$

A generalization of Theorem 4 is:

Theorem 5. Let, on a set $E \subset (-\infty, \infty)$, a real rational function $R_n(x)$ (of order not exceeding n) not exceed the number M in absolute value. Then for any $\delta > 0$ there exists a set

$$E(\delta) = E(\delta, M, R_n)$$

(not depending on the set E) such that $\operatorname{mes} E(\delta) < \delta$ and, for $x \in E \setminus E(\delta)$ and all $p = 0, 1, 2, \ldots$,

$$|R_n^{(p)}(x)| \leq C_p (n/\delta)^p M,$$

where

$$C_p = p!\, 2^{\frac{1}{2}p(p+3)}.$$

For each p the estimate is sharp in order with respect to the totality of n, M, and δ.

By **sharpness in order** here is meant the following: there exist positive constants A_p $(p = 0, 1, 2, \ldots)$ possessing the following properties: for any natural n and p and positive M and δ there will be found a set $E \subset (-\infty, \infty)$ and a real rational fraction $R_n(x)$ of order n such that, whatever set E' of measure $< \delta$ we take, there will be found a point $x \in E \setminus E'$ for which

$$|R_n^{(p)}(x)| > A_p (n/\delta)^{pM}.$$

It turns out that one may take

$$A_p = \frac{1}{1500}\sqrt{p!},$$

and take the function $R_n(x)$ to be one and the same for all p.

2.2. Denote by $G(M, R_n)$ the set of all points z of the complex plane Z for which

$$|R_n(z)| \le M \quad (M > 0)$$

(here $R_n(z)$ has, generally speaking, complex coefficients).

Theorem 6. For any rational function $R_n(z)$ of order not exceeding n and any $\delta > 0$ there exists a set

$$F_1(\delta) = F_1(\delta, M, R_n)$$

such that $\mathrm{mes}_2\, F_1(\delta) < \delta$ and, for

$$z \in G(M, R_n) \setminus F_1(\delta),$$

$$|R_n'(z)| \le \sqrt{\pi/\delta}\, \sqrt{n}\, M.$$

The estimate is sharp for every set of n, M, δ. The equality sign is in fact attained only for

$$R_1(z) = e^{i\alpha} \sqrt{\pi/\delta}\, Mz + b$$

(where $b = \text{const}$, α is real).

The proof follows from the obvious equality

$$\int_{G(M,R_n)} |R_1'(z)|^2\, d\sigma = \pi M^2 n.$$

From the proof it is seen that the last theorem (as, incidentally, Theorem 4 also) generalizes to arbitrary n-sheeted functions. Namely, the following is true:

Theorem 6′. Let in a domain $G \subset Z$ an analytic function $f(z)$ be no more than n-sheeted, and let

$$|f(z)| \le M$$

for $z \in E \subset G$. Then for every $\delta > 0$ there exists a set $F(\delta)$ such that $\mathrm{mes}_2\, F(\delta) < \delta$ and, for

$$z \in E \setminus F(\delta),$$

$$|f'(z)| \le \sqrt{\pi/\delta}\, M \sqrt{n}.$$

A generalization of Theorem 6 is

Theorem 7. Let, on a set E of the complex plane, a rational function $R_n(z)$ (of order n) not exceed the number M in modulus. Then for any $\delta > 0$ there exists a set

$$F(\delta) = F(\delta, M, R_n)$$

(not depending on E) such that $\mathrm{mes}_2\, F(\delta) < \delta$ and, for $z \in E \setminus F(\delta)$ and all $p = 0, 1, 2, \ldots,$

$$|R_n^{(p)}(z)| < \widetilde{C}_p (n/\delta)^{\frac{1}{2}p} M,$$

where

$$\widetilde{C}_p = \sqrt{\pi^p p!}\, 2^{\frac{1}{4}p(p+1)}.$$

The estimate is sharp in order with respect to the totality of n, M, and δ for each fixed p.

Accuracy in the sense of order is understood here in the same sense as in Theorem 5.

2,3. Below we denote by: D_k^0 the unit cube of the k-dimensional real Euclidean space

$$E_k = \{x : x = (x_1, x_2, \dots, x_k)\} \quad (k \geqslant 1)$$

with faces parallel to the coordinate hyperplanes; $_k$ the unit k-cylinder of the complex k-dimensional Euclidean space

$$\widetilde{E}_k = \{z : z = (z_1, z_2, \dots, z_k)\} \quad (k \geqslant 1),$$

i.e.

$$_k = \{z : z = (z_1, z_2, \dots, z_k);\ |z_i| \leqslant 1,\ i = 1, 2, \dots, k\};$$

$R_n(y) = R_n(y_1, y_2, \dots, y_k)$ a rational fraction in the variables $y_1, y_2, \dots, y_k$ of order not exceeding n in each variable. With the aid of Theorems 5 and 7, induction on k proves Theorems 8 and 9.

Theorem 8. Let, on a set $E \subset D_k^0$, the real rational function $R_n(x_1, x_2, \dots, x_k)$ be, in absolute value, not greater than M. Then for any $\delta > 0$ there exists a set $E(\delta) = E(\delta, M, R_n)$ such that $\mathrm{mes}_k\, E(\delta) < \delta$, and for $(x_1, x_2, \dots, x_k) \in E \setminus E(\delta)$ and all $p_i \geqslant 0$ $(i = 1, 2, \dots, k)$, $\sum p_i = p$, the estimate

$$\left| \frac{\partial^p R_n(x_1, x_2, \dots, x_k)}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_k^{p_k}} \right| \leqslant C_{k,p} (n/\delta)^p M,$$

holds, where

$$C_{k,p} = p!\, 2^{1/2} p(p+3)\, 2^{(k-1)p(p+1)}.$$

The estimate is sharp in order jointly in n, M, δ (for $\delta < 1/2$) for every choice of $k, p_1, p_2, \dots, p_k$.

Theorem 9. Let, on a set $\widetilde{E} \subset {}_k$, the function $R_n(z_1, z_2, \dots, z_k)$ be, in modulus, not greater than the number M. Then for any $\delta > 0$ there exists a set $F(\delta) = F(\delta, M, R_n)$ such that $\mathrm{mes}_{2k}\, F(\delta) < \delta$, and for $(z_1, z_2, \dots, z_k) \in \widetilde{E} \setminus F(\delta)$ and all $p_i \geqslant 0$ $(i = 1, 2, \dots, k)$, $\sum p_i = p$, the inequality

$$\left| \frac{\partial^p R_n(z_1, z_2, \dots, z_k)}{\partial z_1^{p_1} \partial z_2^{p_2} \cdots \partial z_k^{p_k}} \right| \leqslant \widetilde{C}_{k,p} (n/\delta)^{\frac{1}{2}pk^2 2^1 4^k p(p+1)} M,$$

is satisfied, where

$$\widetilde{C}_{k,p} = \sqrt{p!}\, \pi^{-\frac{1}{2}pk^2 2^1 4^k p(p+1)}.$$

The estimate is sharp in the sense of order in n, M, δ $(\delta < 1/2)$ for every choice of $k, p_1, p_2, \dots, p_k$.

2,4. By somewhat modifying A. Cartan' s method for estimating from below the modulus of a polynomial (7) (applied in (6a) in estimating the derivative of a rational function of a real variable), one can prove the following theorem:

Theorem 10. For any rational function $R_n(z)$ of order n and any $\delta > 0$ one can find a set $e(\delta, R_n)$ such that $\operatorname{mes}_2 e(\delta, R_n) < \delta$, and for $z \notin e(\delta, R_n)$ and all $p = 0, 1, 2, ...,$

$$\left|R_n^{(p)}(z)\right| \leqslant S_p(n/\sqrt{\delta})^p |R_n(z)|,$$

where

$$S_p = p!\, 2^{\frac{1}{4}p(p+15)}.$$

For each fixed p the estimate is sharp in the sense of order jointly in n, δ.

The example of the function

$$R_n(z) = z^{-n}$$

shows the sharpness of the estimate in the sense of order.

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