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Abstract

Full Text

MATHEMATICS

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ON PERIODICALLY CORRELATED RANDOM SEQUENCES

(Presented by Academician A. N. Kolmogorov, 22 X 1960)

1. Definition. A random sequence x_n , $n = 0, \pm 1, \dots$, is called **periodically correlated** if $\mathbf{M}|x_n|^2 < \infty$ for all n and there exists an integer T such that, for all n and τ ,

$$\mathbf{M}x_n = \mathbf{M}x_{n+T}, \quad \mathbf{M}x_{n+\tau}\bar{x}_n = \mathbf{M}x_{n+T+\tau}\bar{x}_{n+T}. \quad (1)$$

The number T in this case is called the **period of the sequence** x_n .

In the problems considered below one may, without loss of generality, always assume that $\mathbf{M}x_n = 0$. In that case the principal characteristic of the sequence x_n will be its correlation function

$$B(n, \tau) = \mathbf{M}x_{n+\tau}\bar{x}_n, \quad (2)$$

which is periodic in n with period T and therefore admits the representation

$$B(n, \tau) = \sum_{k=0}^{T-1} B_k(\tau) \exp\left(\frac{2\pi i k n}{T}\right). \quad (3)$$

For convenience, extend the functions $B_k(\tau)$, $k = 0, \dots, T-1$, to all integer k by means of the equality $B_k(\tau) = B_{k+T}(\tau)$.

Theorem 1. The function (3) will be the correlation function of some periodically correlated random sequence if and only if the matrix

$$B(\tau) = \|B_{jk}(\tau)\|_{j,k=0,\dots,T-1}, \quad (4)$$

where

$$B_{jk}(\tau) = B_{k-j}(\tau) \exp(2\pi i j/T), \quad (5)$$

is the correlation matrix of some T -dimensional stationary random sequence.

Proof. We have to show that the validity, for arbitrary integers t_1, t_2, \dots, t_n and arbitrary complex a_1, a_2, \dots, a_n , of the inequality

$$\sum_{p,q=1}^n B(t_q, t_p - t_q) a_p \bar{a}_q \geq 0, \quad (6)$$

where $B(t, \tau)$ is the function (3), is equivalent to the validity, for arbitrary k_1, \dots, k_n taking integer values from 0 to $T - 1$, and for the same $t_1, \dots, t_n, a_1, \dots, a_n$ as before, of the inequality

$$\sum_{p,q=1}^n \{B_{k_p k_q}(t_p - t_q)\} a_p \bar{a}_q \geq 0, \quad (7)$$

where $B_{jk}(\tau)$ are the functions (5). Suppose first that (6) holds. In this case, using the fact that, by virtue of (3),

$$B_l(\tau) = \frac{1}{T} \sum_{n=0}^{T-1} B(n, \tau) \exp\left(-\frac{2\pi i n l}{T}\right), \quad (8)$$

we shall have:

$$\begin{aligned} \sum_{p,q} B_{k_p k_q}(t_p - t_q) a_p \bar{a}_q &= \sum_{p,q} B_{k_q - k_p}(t_p - t_q) \exp\left(\frac{2\pi i k_p(t_p - t_q)}{T}\right) a_p \bar{a}_q = \\ &= \frac{1}{T} \sum_{p,q} \exp\left(\frac{2\pi i k_p(t_p - t_q)}{T}\right) a_p \bar{a}_q \sum_{n=0}^{T-1} B(n, t_p - t_q) \exp\left(-\frac{2\pi i n(k_q - k_p)}{T}\right) = \\ &= \frac{1}{T} \sum_{m=0}^{T-1} \left[\sum_{p,q} B(m + t_q, t_p - t_q) a_p \exp\left(\frac{2\pi i k_p(m + t_p)}{T}\right) \times \right. \\ &\quad \left. \times \overline{a_q \exp\left(\frac{2\pi i k_q(m + t_q)}{T}\right)} \right] \geq 0. \end{aligned}$$

If, however, we assume that (7) is satisfied, then, by virtue of (5) and (3),

$$\sum_{p,q} B(t_q, t_p - t_q) a_p \bar{a}_q = \sum_{p,q} a_p \bar{a}_q \sum_{l=0}^{T-1} B_l(t_p - t_q) \exp\left(\frac{2\pi i l t_q}{T}\right) =$$

$$\begin{aligned}
 &= \sum_{p,q} a_p \bar{a}_q \sum_{k=0}^{T-1} B_{k-j}(t_p - t_q) \exp\left(\frac{2\pi i t_q (k-j)}{T}\right) = \\
 &= \frac{1}{T} \sum_{p,q} a_p \bar{a}_q \sum_{i,k} B_{k-j}(t_p - t_q) \exp\left(\frac{2\pi i j (t_p - t_q)}{T}\right) \exp\left(\frac{2\pi i (k t_q - j t_p)}{T}\right) = \\
 &= \frac{1}{T} \sum_{p,q} a_p \bar{a}_q \sum_{j,k} B_{jk}(t_p - t_q) \exp\left(\frac{2\pi i (k t_q - j t_p)}{T}\right) \geq 0,
 \end{aligned}$$

and thus Theorem 1 is proved.

- Using now the spectral representation of the correlation matrix of a multidimensional stationary random sequence, we can formulate Theorem 1 also in the following form:

In order that the function (3) be the correlation function of some periodically correlated sequence, it is necessary and sufficient that the functions $B_k(\tau)$, $k = 0, \dots, T-1$, all be representable in the form of Fourier-Stieltjes integrals

$$B_k(\tau) = \int_0^{2\pi} e^{i\tau\lambda} dF_k(\lambda), \quad (9)$$

where $F_k(\lambda)$, $k = 0, \dots, T-1$, are complex functions of bounded variation, continuous from the left, such that for any λ_1 and $\lambda_2 \geq \lambda_1$ the increment $\mathcal{F}(\lambda_2) - \mathcal{F}(\lambda_1)$ of the matrix

$$\mathcal{F}(\lambda) = \|F_{k-j}((\lambda - 2\pi j)/T)\|_{j,k=0,\dots,T-1} \quad (10)$$

(for $k < 0$ and for $\lambda < 0$ or $\lambda > 2\pi$ the functions $F_k(\lambda)$ are extended by means of the equalities $F_k(\lambda) = F_{k+T}(\lambda)$, $F_k(\lambda + 2\pi) = F_k(\lambda) + F_k(2\pi)$, $F_k(0) = 0$ for all k) be a Hermitian non-negative definite matrix.

Let us also note that from formulas (3) and (9) it follows easily that the periodically correlated sequence x_n will necessarily be harmonizable in the sense of (1), i.e., will admit the representation

$$x_n = \int_0^{2\pi} e^{in\lambda} z(d\lambda), \quad (11)$$

where $z(\Lambda)$ is a random function of a set with zero mathematical expectation, satisfying the condition

$$\mathbf{M}z(\Lambda)\overline{z(M)} = \sum_{k=-T+1}^{T-1} \int_{\Lambda \cap (M - \frac{2\pi k}{T})} dF_k(\lambda) \quad (12)$$

($M - a$ is the set of points $\mu - a$, $\mu \in M$).

This can otherwise be expressed as follows: the spectrum of the random sequence x_n is concentrated on $2T - 1$ straight-line segments $\lambda - \mu = 2\pi k/T$, $k = -T + 1, \dots, T - 1$, lying inside the square $0 \leq \lambda, \mu < 2\pi$, and the functions $F_k(\lambda)$ specify the spectral intensities on these $2T - 1$ lines. From this, of course, there also follows the condition formulated above concerning the matrix $\mathfrak{F}(\lambda)$.

Formulas (11) and (12) also follow from the possibility of representing the periodically correlated sequence x_n in the form

$$x_n = \sum_{k=0}^{T-1} z_n^k \exp\left(\frac{2\pi i k n}{T}\right), \quad (13)$$

where $z_n = (z_n^0, \dots, z_n^{T-1})$ is a T -dimensional stationary (in the broad sense) sequence given by the formula

$$z_n^k = \frac{1}{T} \sum_{m=0}^{T-1} y_{\frac{n-m}{T}}^m \exp\left(-\frac{2\pi i m k}{T}\right),$$

and $y_t = (y_t^0, \dots, y_t^{T-1})$ is a T -dimensional random process obtained by a formal substitution, in the spectral expansion of the T -dimensional stationary sequence

$$y_n = (x_{nT}, x_{nT+1}, \dots, x_{nT+T-1}),$$

of the discrete parameter n by the continuous parameter t .

Remark. The case of multidimensional periodically correlated random sequences

$$x_n = (x_n^1, \dots, x_n^s),$$

for which

$$\mathbf{M}x_n^l = \mathbf{M}x_{n+T}^l, \quad \mathbf{M}x_{n+\tau}^l \overline{x_n^m} = \mathbf{M}x_{n+T+\tau}^l \overline{x_{n+T}^m}. \quad (14)$$

can be treated analogously. Assuming here also that $\mathbf{M}x_n^k \equiv 0$, we must take as the basic statistical characteristics of such a multidimensional sequence the correlation functions

$$\mathbf{M}x_{n+\tau}^l \overline{x_n^m} = B^{lm}(n, \tau) = \sum_{k=0}^{T-1} B_k^{lm}(\tau) \exp\left(\frac{2\pi i k n}{T}\right). \quad (15)$$

In order that the functions (15) be the correlation functions of some multidimensional T -periodic random sequence, it is necessary and sufficient that the $(sT \times sT)$ -matrix

$$B(\tau) = \|B_{lm}(\tau)\|_{l,m=1,\dots,s}, \quad B_{lm}(\tau) = \|B_{jk}^{lm}(\tau)\|_{j,k=0,\dots,T-1},$$

where

$$B_{jk}^{lm}(\tau) = B_{k-j}^{lm}(\tau) \exp\left(\frac{2\pi i j \tau}{T}\right),$$

be the correlation matrix of some sT -dimensional stationary random sequence. Otherwise this can be formulated as follows: the functions $B_k^{lm}(\tau)$ must be representable in the form

$$B_k^{lm}(\tau) = \int_0^{2\pi} e^{i\tau\lambda} dF_k^{lm}(\lambda),$$

where the left-continuous complex functions of bounded variation $F_k^{lm}(\lambda)$ possess the property that the $(sT \times sT)$ -matrix

$$\mathfrak{F}(\lambda) = \|F_{lm}(\lambda)\|_{l,m=1,\dots,s}, \quad F_{lm}(\lambda) = \left\| F_{k-j}^{lm} \left(\frac{\lambda - 2\pi j}{T} \right) \right\|_{j,k=0,\dots,T-1}$$

is a matrix with Hermitian nonnegative-definite increments.

The further results of the present paper can also without difficulty be carried over to the case of multidimensional periodically correlated sequences; however, for simplicity of notation we shall henceforth consider only the one-dimensional case.

3. It is easy to see that the random sequence x_n , $n = 0, \pm 1, \dots$, will be periodically correlated with period T if and only if the T -dimensional random sequence $y_n = (y_n^0, \dots, y_n^{T-1})$, where $y_n^k = x_{nT+k}$, $-\infty < n < \infty$, $k = 0, \dots, T-1$, is stationary in the wide sense, i.e.,

$$\mathbf{M} y_{n+\tau}^j \overline{y_n^k} = \int_0^{2\pi} e^{i\tau\lambda} dF_{jk}(\lambda), \quad j, k = 0, \dots, T-1. \quad (16)$$

Denote by H_x , $H_x(n)$, and $\hat{H}_x(n)$ the linear closures (in the mean square), respectively, of the random variables $\{x_\tau, -\infty < \tau < \infty\}$, $\{x_\tau, \tau \leq n\}$, and $\{x_\tau, \tau \neq n\}$. Following (2), we shall call a random sequence x_n **regular** if $\bigcap_n H_x(n) = 0$, and **singular** if $\bigcap_n H_x(n) = H_x$. A regular sequence x_n for which, for all n , $x_n \notin H_x(n-1)$, will be called **completely regular**. We shall say that the sequence x_n is **minimal** if

$$\min_{h \in \hat{H}_x(n)} \mathbf{M} |x_n - h|^2 = \sigma_n^2 > 0$$

for all n . In this case the condition of complete regularity of the sequence x_n obviously coincides with the condition of regularity of highest rank (in the sense

of (3)) of the multidimensional stationary sequence y_n , while the condition of minimality of the sequence x_n coincides with the condition of minimality of highest rank (in the sense of (4,5)) of the sequence y_n .

Theorem 2. The matrix $\|dF_{jk}(\lambda)/d\lambda\|_{j,k=0,\dots,T-1}$ is determined by the equality

$$\left\| \frac{dF_{jk}(\lambda)}{d\lambda} \right\|_{j,k=0,\dots,T-1} = TU^{-1}(\lambda) \frac{d\mathcal{F}(\lambda)}{d\lambda} U(\lambda) \quad \text{for almost all } \lambda, \quad (17)$$

where $d\mathcal{F}(\lambda)/d\lambda$ is the matrix of the derivatives of all matrix elements of the matrix $\mathcal{F}(\lambda)$, and $U(\lambda) = \|U_{jk}(\lambda)\|_{j,k=0,\dots,T-1}$ is a unitary matrix depending on λ , with elements

$$U_{jk}(\lambda) = T^{-1/2} \exp\left(\frac{2\pi ijk - ik\lambda}{T}\right).$$

From Theorem 2, by virtue of the results of (3-5), the following propositions follow:

Theorem 3. A periodically correlated random sequence x_n is completely regular if and only if: 1) all $F_l(\lambda)$ are absolutely continuous; 2) $\det d\mathcal{F}(\lambda)/d\lambda > 0$ almost everywhere; 3)

$$\int_0^{2\pi} \ln \det \frac{d\mathcal{F}(\lambda)}{d\lambda} d\lambda > -\infty.$$

Theorem 4. A periodically correlated random sequence x_n is minimal if and only if: 1) $\det d\mathcal{F}(\lambda)/d\lambda > 0$ almost everywhere; 2)

$$\int_0^{2\pi} \text{Sp} \left(\frac{d\mathcal{F}(\lambda)}{d\lambda} \right)^{-1} d\lambda < \infty.$$

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CITED LITERATURE

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