ON THE CONTINUITY OF THE NEMYTSKII OPERATOR IN ORLICZ SPACES

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Abstract

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I. V. SHRAGIN

ON THE CONTINUITY OF THE NEMYTSKII OPERATOR IN ORLICZ SPACES

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In papers ([1−6]) certain sufficient conditions were established for the continuity of the Nemytskii operator in Orlicz spaces. However, even the identity transformation of an arbitrary Orlicz space (such a transformation is a special case of a Nemytskii operator) does not satisfy these conditions.

Here we establish necessary and sufficient conditions for the continuity of the Nemytskii operator in Orlicz spaces, as well as a number of other propositions.

1°. **Some information from the theory of Orlicz spaces in the sense of Zaanen–Luxemburg** ([7,8]). Let a nonnegative function $\varphi(t)$, $0 \leq t < \infty$ (which may assume infinite values, but does not become identically on $[0, +\infty)$ either zero or infinity), be nondecreasing, with $\varphi(t+0) = \varphi(t)$ for every t.

The function

$$\Phi(u) = \int_0^u \varphi(t)\, dt, \quad u \geqslant 0,$$

is called a Young function. Put, for $\alpha > 0$,

$$L_\Phi^\alpha = \left\{ u(x) : \int_B \Phi[\alpha |u(x)|]\, dx < \infty \right\},$$

where B is some subset of finite-dimensional Euclidean space of positive (finite or infinite) Lebesgue measure. By definition, the Orlicz space L^Φ is the union of all classes L_Φ^α. We shall consider it normed by means of the norm

$$\|u\|_\Phi = \inf \left\{ \rho > 0 : \int_B \Phi[\rho^{-1} |u(x)|]\, dx \leqslant 1 \right\}.$$

By L_Φ^f is denoted the subspace of the space L^Φ that is the intersection of all classes L_Φ^α. We note that: 1) L_Φ^f consists only of the zero element if and only if $d_\Phi = \sup\{u : \Phi(u) < \infty\} < \infty$; 2) $L_\Phi^f = L^\Phi$ if and only if $d_\Phi = \infty$ and $\Phi(u)$ satisfies the Δ_2-condition: $\Phi(2u) \leqslant C\Phi(u)$ for $u \geqslant u_0 \geqslant 0$ ($C = \text{const}$), where $u_0 = 0$ if $\operatorname{mes} B = \infty$.

In this note we shall also consider Orlicz spaces whose elements are vector functions defined on the set B. Namely, let $M_1(u), M_2(u), \dots, M_s(u)$ be arbitrary

Young functions. Denote by $\vec{L}_M^{\vec{\alpha}}$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s)$, the class of vector functions $\mathbf{u}(x) = (u_1(x), u_2(x), \dots, u_s(x))$, for which $u_k(x) \in L_{M_k}^{\alpha_k}$, $k = 1, 2, \dots, s$. Put further,

$$L^M = \{u(x) : u_k(x) \in L^{M_k},\ k = 1, 2, \dots, s\}, \quad L_M^f = \{u(x) : u_k(x) \in L_{M_k}^f,\ k = 1, 2, \dots, s\}.$$

In the space L^M we introduce the norm

$$\|u\|_M = \max \|u_k\|_{M_k}, \quad k = 1, 2, \dots, s.$$

2°. Let the real-valued function $g(u_1, u_2, \dots, u_s, x) \equiv g(\mathbf{u}, x)$, where $x \in B$, $u_k \in (-\infty, +\infty)$, $k = 1, 2, \dots, s$, be measurable in x for each $\mathbf{u}$ and continuous in $\mathbf{u}$ for almost every $x \in B$. Such a function generates a Nemytskii operator h, $h\mathbf{u}(x) = g(\mathbf{u}(x), x)$.

In the present section we shall establish several propositions on the action of the operator h in Orlicz spaces. The proofs of these propositions use the construction applied in ([3],[4]).

Theorem 1. *In order that the operator h act from $L_M^{\vec{\alpha}}$ to L_Φ^β, it is necessary and sufficient that there exist a $\gamma > 0$ and an integrable function $f(x)$ on B such that, for all $x \in B$ and $\mathbf{u} \in R^s$,*

$$\Phi[\beta|g(\mathbf{u}, x)|] \le \gamma \sum_{k=1}^{s} M_k[\alpha_k |u_k|] + f(x).$$

Put $\mathbf{T}_r = \{\mathbf{u} \in L^M : \|\mathbf{u}\|_M \le r\}$. It is known ([6],[8]) that $\mathbf{T}_r \subset L_M^{\vec{\alpha}}$, where $\vec{\alpha} = (r^{-1}, r^{-1}, \dots, r^{-1})$.

Theorem 2. *If h maps the ball $\mathbf{T}_r$ into the class L_Φ^β (in the space L^Φ), then it maps $L_M^{\vec{\alpha}}$, where $\vec{\alpha} = (r^{-1}, r^{-1}, \dots, r^{-1})$, into L_Φ^β (respectively, into some class L_Φ^μ).*

Theorem 3. *If $d_{M_k} = \infty$, $k = 1, 2, \dots, s$, and h maps the ball $\{\mathbf{u} \in L_M^f : \|\mathbf{u}\|_M \le r\}$ into the class L_Φ^β (in the space L^Φ), then it maps some class $L_M^{\vec{\lambda}}$ into L_Φ^β (respectively, into some class L_Φ^μ).*

We note that the last two theorems contain assertions close to Theorem 17.2 of ([6]).

3°. **Criterion for continuity of the Nemytskii operator at a point.** In the present section we assume throughout that $h\mathbf{u}_0 \in L^\Phi$, where $\mathbf{u}_0$ is a fixed point of the space L^M. By continuity of the operator h at the point $\mathbf{u}_0$ we shall mean the following: h maps some neighborhood U (in the space L^M) of the point $\mathbf{u}_0$ into the space L^Φ, and $\lim \|h\mathbf{u} - h\mathbf{u}_0\|_\Phi = 0$ when $\|\mathbf{u} - \mathbf{u}_0\|_M \to 0$, where $\mathbf{u} \in U$.

Lemma 1. *For continuity of h at the point $\mathbf{u}_0$ it is necessary and sufficient that, for every $\mu > 0$, there exist a $\vec{\lambda}$ such that the operator h_1, $h_1\mathbf{v} = h(\mathbf{u}_0+\mathbf{v}) - h\mathbf{u}_0$, maps $L_M^{\vec{\lambda}}$ into L_Φ^μ.*

Lemma 1′. *For continuity of h at the point $\mathbf{u}_0$ it is necessary, and if $d_{M_k} = \infty$, $k = 1, 2, \dots, s$, then also sufficient, that the operator h_1 map L_M^f into L_Φ^f.*

Theorem 4. *For continuity of h at the point $\mathbf{u}_0$ it is necessary and sufficient that, for every $\mu > 0$, there exist a $\rho > 0$ and an integrable function $f(x)$ on B such that*

$$\Phi[\mu\,|g(\mathbf{u}_0(x)+\mathbf{v}, x) - g(\mathbf{u}_0(x), x)|] \le \sum_{k=1}^{s} M_k[\rho|v_k|] + f(x)$$

for all $x \in B$ and $\mathbf{v} \in R^s$.

Remark. Let $d_{M_k} = \infty$, $k = 1, 2, \dots, s$. Then continuity (in the sense indicated above) of the operator h at the point $\mathbf{u}_0$ follows from continuity in the "narrow" sense: $\lim \|h(\mathbf{u}_0+\mathbf{v}) - h\mathbf{u}_0\|_\Phi = 0$ as $\|\mathbf{v}\|_M \to 0$, where $\mathbf{v} \in L_M^f$.

4°. With the aid of the criterion for continuity of h at a point established above, the following propositions are obtained.

Theorem 5. If h is continuous at the point $\mathbf{u}_0 \in L^M$, then it is continuous at every point $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$, where $\mathbf{u}_1 \in L_M^f$.

Corollary 1. If $L^M = L_M^f$ and h is continuous at some point $\mathbf{u}_0 \in L^M$, then it is continuous at every point $\mathbf{u} \in L^M$.

Theorem 6. If h is continuous at every point of the ball T_r, then it is continuous at every point of the class $L_M^{\vec{\alpha}}$, where $\vec{\alpha} = (r^{-1}, r^{-1}, \dots, r^{-1})$.

5°. Criterion for the continuity of the Nemytskii operator on the subspace L_M^f.

Theorem 7. Let $g(0, x) \in L^\Phi$. Then, for h to be continuous at every point $\mathbf{u}_0 \in L_M^f$, it is necessary and sufficient that for every $\mu > 0$ there exist $\rho > 0$ and a function $f(x)$, integrable on B, such that for all $x \in B$ and $\mathbf{v} \in R^s$

$$\Phi\,[\mu\,|g(\mathbf{v}, x) - g(0, x)|] \ge \sum_{k=1}^{s} M_k[\rho\,|v_k|] + f(x).$$

Theorem 7′. Let $g(0, x) \in L^\Phi$. Then, for h to be continuous at every point $\mathbf{u}_0 \in L_M^f$, it is necessary, and if $d_{M_k} = \infty$, $k = 1, 2, \dots, s$, also sufficient, that the operator $\tilde{h}$,

$$\tilde{h}\mathbf{v}(x) = g(\mathbf{v}(x), x) - g(0, x),$$

act from L_M^f into L_Φ^f.

6°. Criterion for the continuity of the Nemytskii operator on the class $L_M^{\vec{\alpha}}$ and the space L^M.

Theorem 8. Let h act from $L_M^{\vec{\alpha}}$ into L^Φ (from L^M into L^Φ). Then, for h to be continuous at every point $\mathbf{u}_0 \in L_M^{\vec{\alpha}}$ (respectively, $\mathbf{u}_0 \in L^M$), it is necessary and sufficient that for every $\mu > 0$ (respectively, for any positive $\alpha_1, \alpha_2, \dots, \alpha_s$ and μ) there exist $\gamma > 0$, $\rho > 0$, and a function $f(x)$, integrable on B, such that

$$\Phi\left[\mu\left|g(\mathbf{u}+\mathbf{v}, x) - g(\mathbf{u}, x)\right|\right] \leq \gamma \sum_{k=1}^{s} M_k[\alpha_k\left|u_k\right|] + \sum_{k=1}^{s} M_k[\rho\left|v_k\right|] + f(x)$$

for all $x \in B$ and $\mathbf{u}, \mathbf{v} \in R^s$.

Kostroma State Pedagogical Institute
named after N. A. Nekrasov

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Note: Figure translations are in progress. See original paper for figures.