

# BOUNDARY-VALUE PROBLEMS WITH CONDITIONS AT $(-\infty, \infty)$

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**Abstract**

**Full Text**

**MATHEMATICS**

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**BOUNDARY-VALUE PROBLEMS WITH  
CONDITIONS AT  $\pm\infty$**

*(Presented by Academician I. G. Petrovskii on 18 III 1961)*

In this note we consider the boundary-value problem

$$\ddot{x} + f(t, x, \dot{x})\dot{x} = 0; \tag{1}$$

$$x(-\infty) = a, \quad x(+\infty) = b, \tag{1'}$$

where  $a$  and  $b$  are given numbers, and the analogous problem for the autonomous third-order equation

$$\ddot{x} + \varphi(x, \dot{x}, \ddot{x})\ddot{x} = 0; \tag{2}$$

$$\dot{x}(-\infty) = a, \quad x(0) = b, \quad \dot{x}(+\infty) = c. \tag{2'}$$

Such problems are of interest in theoretical and applied questions <sup>(1-3)</sup>. We shall assume that the function  $f(t, x, y)$  is defined in the whole space  $(t, x, y)$ , is continuous in  $t$ , and satisfies a Lipschitz condition with respect to the variables  $x, y$  in every bounded domain of the space. With respect to the function  $\varphi(x, y, z)$  we shall assume that it satisfies a Lipschitz condition in every bounded domain of the space  $(x, y, z)$ .

First of all, let us say a few words in justification of the special form of equation (1). For the autonomous equation  $\ddot{x} = f(x, \dot{x})$ , if a solution of problem (1') exists, then it is certainly not unique, since  $x(t + C)$ , for any  $C$ , will also be a solution of this problem.

On the contrary, for the equation  $\ddot{x} = f(t, \dot{x})$ , as is easily verified, if a solution of the problem exists, then it is unique. Finally, for the equation  $\dot{x} = \varphi(t, x)$  a solution of such a problem exists only as an exception. Thus, for example, the equation  $\ddot{x} = a(t)x$ ,  $a(t) \geq 0$ , has a solution only if  $a = b = 0$ . In view of all that has been said, we shall confine ourselves to equation (1) of the special form. The same considerations may also be adduced in justification of the special form of equation (2).

**Theorem 1.** Let the function  $f(t, x, y)$  satisfy the conditions:

A. For each bounded domain  $D$  of the plane  $(x, y)$  there exist numbers  $\sigma > 0$  and  $T > 0$  such that  $tf(t, x, y) \geq 1 + \sigma$  for  $|t| \geq T$  and  $(x, y) \in D$ .

B.  $|f(t, x, y)y| \leq c(t, x)(1 + y^2)$ , where  $c(t, x) \geq 0$  is continuous for  $t^2 + x^2 < \infty$ ,  $-\infty < y < \infty$ .

Then a solution of problem (1), (1') exists.

**Proof.** Since the function  $f$  satisfies condition B, the boundary-value problem

$$\ddot{x} + f(t, x, \dot{x})\dot{x} = 0; \quad (1)$$

$$x(-n) = a, \quad x(n) = b \quad (1'')$$

has at least one solution for every integer  $n \geq 1$ . Denote one of these solutions by  $x_n(t)$  ( $n = 1, 2, \dots$ ). Since no solution  $x(t)$  of equation (1) can have an extremum, the functions  $x_n(t)$  are bounded-

are, in aggregate,  $a \leq x_n(t) \leq b$ ,  $|t| \leq n$ . (We assume that  $a < b$ ; the case  $a > b$  is treated analogously.) Now take the domain  $D$  in the form of the square  $a \leq x \leq b$ ,  $0 \leq y \leq b - a$ . For this domain choose numbers  $\sigma > 0$  and  $T > 0$  such that  $tf(t, x, y) > 1 + \sigma$  for  $|t| \geq T$ .

Without loss of generality we may suppose that  $\sigma < 1$ , and  $T > 1$ . We now construct a solution  $u(t)$  of the equation

$$\ddot{u} + \frac{1 + \sigma}{t}\dot{u} = 0$$

satisfying the conditions  $u(T) = a$ ,  $u(\infty) = b$ . It has the form  $u(t) = b - (b - a)(T/t)^\sigma$ , so that

$$\dot{u}(t) = \sigma(b - a)T^\sigma/t^{1+\sigma}.$$

Obviously,  $0 < \dot{u}(t) < b - a$  for  $t \geq T$ . Let  $n > T$ . We shall show that the  $x_n(t)$  ( $T \leq t \leq n$ ) do not, for any  $n$  and  $t$ , intersect the graph of  $u(t)$ . Indeed, if this were so, then at the point of intersection  $t = t_0$  we would have  $u(t_0) = x_n(t_0)$  and  $\dot{u}(t_0) \geq \dot{x}_n(t_0)$ . But then, taking this point as initial and noting that the point  $(x_n(t_0), \dot{x}_n(t_0)) \in D$ , where inequality A is satisfied, we obtain

$$\dot{x}_n(t) = \dot{x}_n(t_0) \exp \left[ - \int_{t_0}^t f(\tau, x_n(\tau), \dot{x}_n(\tau)) d\tau \right] < \dot{u}(t_0) \exp \left[ - \int_{t_0}^t \frac{1 + \sigma}{\tau} d\tau \right] = \dot{u}(t_0) \left( \frac{t_0}{t} \right)^{1+\sigma}$$

for  $t \geq t_0$ , and hence  $x_n(t)$  cannot intersect the graph of  $u(t)$  from below upward, which contradicts the condition  $x_n(n) = b > u(n)$ . Consequently, the graphs of  $x_n(t)$  lie between the graph of  $u(t)$  and the straight line  $x = b$ . For  $t \leq -T$  one

can analogously show that  $x_n(t)$  lies between the straight line  $x = a$  and the graph  $v(t)$  of the solution of the problem

$$\ddot{v} + \frac{1 + \sigma}{t} \dot{v} = 0, \quad v(-\infty) = a, \quad v(T) = b.$$

Since the sequence  $\{x_n(T)\}$  is bounded, the sequence  $\{\dot{x}_n(T)\}$ , by condition B, will also be bounded. Now from the sequence  $\{x_n(T), \dot{x}_n(T)\}$  we choose a convergent subsequence so that, as  $k \rightarrow \infty$ ,

$$x_{n_k}(T) \rightarrow \alpha, \quad \dot{x}_{n_k}(T) \rightarrow \beta.$$

The solution of equation (1) satisfying the conditions  $x(T) = \alpha$ ,  $\dot{x}(T) = \beta$  will be the desired one. Indeed, it is obvious that  $u(t) \leq x(t) \leq b$  ( $t \geq T$ ), and since  $u(t) \rightarrow b$  as  $t \rightarrow +\infty$ , we have  $x(+\infty) = b$ . Similarly, for  $t \leq -T$  we have  $a \leq x(t) \leq v(t)$ , and since  $v(t) \rightarrow a$  as  $t \rightarrow -\infty$ , it follows that  $x(-\infty) = a$ , as was required to prove.

**Remark.** In the general case the number  $\sigma$  cannot be replaced by zero, as is shown by the example of the equation

$$\ddot{x} + \frac{t}{1 + t^2} \dot{x} = 0.$$

Then

$$x(t) = c_1 + c_2 \ln(t + \sqrt{1 + t^2}),$$

and for  $c_2 \neq 0$ ,  $x(\infty) = \infty$ .

It is also easy to construct examples showing that the exponent 2 in condition B cannot be replaced by  $2 + \varepsilon$ .

**Theorem 2 (uniqueness theorem).** *If  $f(t, x, y)$  is a nondecreasing function of  $x$  for  $y > 0$ , then the solution of problem (1), (1') is unique for  $a < b$ . If  $f(t, x, y)$  is a nonincreasing function of  $x$  for  $y < 0$ , then the solution of the problem is unique for  $a > b$ . Finally, if  $f(t, x, y)$  is a nondecreasing function of  $x$  for all  $y$ , then the solution is unique for arbitrary  $a$  and  $b$ .*

**Proof.** Suppose there exist two solutions of the problem,  $x_1(t)$  and  $x_2(t)$ . Then their difference  $x_2(t) - x_1(t) = u(t)$  satisfies the condition

$$u(\pm\infty) = 0$$

and the equation

$$\ddot{u} + \dot{u} \frac{f(t, x_1, \dot{x}_2) \dot{x}_2 - f(t, x_1, \dot{x}_2) \dot{x}_2}{\dot{x}_2 - \dot{x}_1} + u \frac{\dot{x}_2 [f(t, x_2, \dot{x}_2) - f(t, x_1, \dot{x}_2)]}{x_2 - x_1} = 0,$$

whence it follows immediately that  $u(t) \equiv 0$ , since the equation  $\ddot{u} + a(t)\dot{u} + b(t)u = 0$  for  $b(t) \leq 0$  can have neither a positive maximum nor a negative minimum, as was required to prove.

Let us now turn to problem (2), (2'), considering first the case  $ac > 0$ , and then the case  $ac < 0$ . Let, for example,  $a > 0$ ,  $c > 0$ . From equation (2) it is clear that, for any solution,  $\dot{x}(t)$  does not change sign. Therefore  $x(t)$  varies monotonically and, hence,  $\dot{x}(t) \geq \min(a, c) > 0$  for  $-\infty < t < \infty$  for any solution of problem (2), (2'). Make the substitution  $\dot{x} = u(x)$ ,  $-\infty < x < \infty$ . Then  $\ddot{x} = uu'$ ,  $\ddot{x} = u''u^2 + u'^2u$  (the prime denotes differentiation with respect to  $x$ ).

Substituting into equation (2), we obtain

$$u'' + \frac{u' + \varphi(x, u, uu')}{u} u' = 0; \quad (3)$$

$$u(-\infty) = a, \quad u(+\infty) = c. \quad (3')$$

Thus, any solution of problem (2), (2') is a solution of problem (3), (3'), which we have already studied. Having solved problem (3), (3') and found  $u(x)$ , we obtain a solution  $x(t)$  from the substitution  $dx/dt = u(x)$ , using the condition  $x(0) = b$ ; moreover, if problem (3), (3') has a unique solution, then problem (2), (2') also has a unique solution. From these considerations, in particular, there follows the existence and uniqueness of the solution of problem (2') for the Blasius equation  $\ddot{x} + 2x\dot{x} = 0$ , which is of interest in the study of the problem of the mixing of two gas jets <sup>(3)</sup>.

Let us now consider the case when  $a$  and  $c$  have different signs, for example  $a < 0$ ,  $c > 0$ . Then, if a solution  $x(t)$  exists, it will necessarily have a minimum at some  $t = t_0$ ,  $\dot{x}(t_0) = 0$ , and if  $t_0 \neq 0$ , then problem (2), (2') has two solutions  $x_1 = x(t + t_1)$  and  $x_2 = x(t + t_2)$ , where  $t_1$  and  $t_2$  are the two roots of the equation  $x(t) = b$ . If  $t_0 = 0$ , then already for  $b + \varepsilon$  ( $\varepsilon > 0$ ) there will exist two solutions of the problem.

Thus, in the case  $ac < 0$ , uniqueness of the solution of problem (2), (2'), generally speaking, does not hold.

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