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Abstract

Full Text

Physics

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ON THE QUESTION OF THE STATE OF A GAS MOVING NEAR A SOLID SURFACE

The state of a gas near a solid surface of flow differs substantially from the bulk state. Although the boundary effect is of order λ/a , where λ is the mean free path of the gas molecules and a is the characteristic size of the problem, nevertheless at any pressure there exists a boundary layer with thickness of order λ (the so-called “Knudsen layer”), in which it plays a noticeable role.

Maxwell ⁽¹⁾ was the first, as he himself called it, to carry out a preliminary analysis of this problem. Subsequently this problem came within the purview of numerous investigators. Essentially the first attempt to find rigorously the distribution function of gas molecules in the boundary layer, without which it is impossible to find the exact value of the slip coefficient needed for calculating the resistance to the motion of a sphere and for other problems, was the work of Gross and others ^(2,3), who considered the Couette flow of a gas between two parallel plates. However, the results of this work contain errors that are of fundamental importance and that arose as a result of incorrect numerical calculations. Moreover, the solution of the problem proposed in ^(2,3) does not satisfy the most important physical requirement: at a sufficiently large distance between the plates and at a sufficient distance from each of them (in the interval between the plates), the coefficients in the distributions of the velocities of molecules flying in opposite directions along the normal to the plates must be identical. In the works ^(2,3), however, the distribution functions for the incident and ascending fluxes differ substantially even midway between the plates, at any distance between the latter.

Below we shall consider the plane flow of a gas bounded by one wall, using the method developed in ^(2,3).

The distribution function of gas molecules satisfies the Boltzmann equation. We shall seek it in the form

$$f = f_0[1 + \Phi(\mathbf{c}, x)], \quad (1)$$

where $\mathbf{c} = (m/2kT)^{1/2}\mathbf{v}$; \mathbf{v} is the velocity of a molecule, m its mass, T the absolute temperature; k is Boltzmann's constant; x is the distance measured from the wall along the normal to it; $f_0 = n(m/2\pi kT)^{3/2} \exp(-c^2)$. The correction

Φ to the distribution function in the stationary state is found from the solution of the equation

$$c_x \partial \Phi / \partial x = J(\Phi), \quad (2)$$

where $J(\Phi)$ is the collision integral.

As the gas-kinetic boundary condition at the solid wall, for simplicity we take the law of diffuse reflection. In the bulk the gas is described by the Chapman–Enskog distribution (4), which can be written in the form

$$f = f_0 \left(1 + \frac{m}{kT} v_z q_z - b_1 \frac{m}{2kT} \frac{v_z v_x}{n} \frac{\partial q_z}{\partial x} \right),$$

where q_z is the velocity of the center of gravity of the gas; n is the number of gas molecules per unit volume; b_1 is a constant depending on

law of interaction of gas molecules. For hard spheres $b_1 = \frac{5}{4} \sqrt{\pi n} \lambda (m/2kT)^{1/2}$.

It is expedient to introduce two distribution functions $f^+(\mathbf{v}, x)$ for molecules with $v_x > 0$ and $f^-(\mathbf{v}, x)$ for molecules with $v_x < 0$, so that the entire set of molecules is described by the distribution function $f(\mathbf{v}, x) = f^+(\mathbf{v}, x) + f^-(\mathbf{v}, x)$; $f^+(\mathbf{v}, x) = 0$ for $v_x < 0$; $f^-(\mathbf{v}, x) = 0$ for $v_x > 0$.

The “boundary” conditions for f^+ and f^- thus have the form $f^+(\mathbf{v}, 0) = f_0$,

$$f^-(\mathbf{v}, d) = f_0 \left(1 + \frac{m}{kT} v_z q_z(d) - b_1 \frac{m}{2kT} \frac{v_x v_z}{n} \frac{\partial q_z}{\partial x} \right), \quad \text{where } d \gg \lambda.$$

Introducing the notation $u_z(x) = (m/2kT)^{1/2} q_z(x)$, $l = \frac{5}{4} \sqrt{\pi} \lambda$, and passing to dimensionless velocities, we obtain

$$\Phi^-(\mathbf{c}, d) = 2c_z u_z(d) - l c_x c_z \frac{\partial u_z}{\partial x}(d), \quad \Phi^+(\mathbf{c}, 0) = 0. \quad (3)$$

We shall seek the solution of equation (2) with the boundary conditions (3) in the form

$$\Phi^\pm(\mathbf{c}, x) = a_0^\pm(x) c_z + a_1^\pm(x) c_x c_z. \quad (4)$$

The conditions (3) will be satisfied if

$$a_0^+(0) = 0, \quad a_1^+(0) = 0, \quad a_0^-(d) = 2u_z(d), \quad a_1^-(d) = -l \frac{\partial u_z}{\partial x}(d). \quad (5)$$

It is convenient to introduce the auxiliary function $\text{sign } c_x = +1$ for $c_x > 0$; $\text{sign } c_x = -1$ for $c_x < 0$. With its aid Φ is expressed in terms of Φ^+ and Φ^- as follows:

$$\begin{aligned} \Phi &= \Phi^+ \frac{1 + \text{sign } c_x}{2} + \Phi^- \frac{1 - \text{sign } c_x}{2} = \\ &= \left(\frac{a_0^+ + a_0^-}{2} + \frac{a_1^+ + a_1^-}{2} c_x \right) c_z + \left(\frac{a_0^+ - a_0^-}{2} + \frac{a_1^+ - a_1^-}{2} c_x \right) c_z \text{sign } c_x. \end{aligned} \quad (6)$$

Since $J(\Phi)$ is a linear operator, then

$$\begin{aligned} J(\Phi) &= \frac{a_0^+ + a_0^-}{2} J(c_z) + \frac{a_1^+ + a_1^-}{2} J(c_x c_z) + \\ &+ \frac{a_0^+ - a_0^-}{2} J(c_z \text{sign } c_x) + \frac{a_1^+ - a_1^-}{2} J(c_x c_z \text{sign } c_x). \end{aligned} \quad (7)$$

According to the law of conservation of momentum in collisions, $J(c_z) = 0$. Finally, multiplying successively equalities (2), taking account of (6), by $c_z(1 \pm \text{sign } c_x)e^{-c^2} d\vec{c}$ and $c_x c_z(1 \pm \text{sign } c_x)e^{-c^2} d\vec{c}$ and integrating with respect to \mathbf{c} from $-\infty$ to $+\infty$, we obtain a system of equations for determining $a_0^\pm(x)$ and $a_1^\pm(x)$:

$$\begin{aligned} \frac{d}{dx} \left(\pm a_0^\pm + a_1^\pm \frac{\sqrt{\pi}}{2} \right) &= \pm \frac{a_1^+ + a_1^-}{\pi} I_2 \pm \frac{a_0^+ - a_0^-}{\pi} I_1, \\ \frac{d}{dx} \left(a_0^\pm \frac{\sqrt{\pi}}{2} \pm a_1^\pm \right) &= \frac{a_0^+ - a_0^-}{\pi} I_2 + \frac{a_1^+ + a_1^-}{\pi} I_3 \pm \frac{a_1^+ - a_1^-}{\pi} I_4, \end{aligned} \quad (8)$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} c_z \text{sign } c_x J(c_z \text{sign } c_x) e^{-c^2} d\vec{c}, & I_2 &= \int_{-\infty}^{\infty} c_z \text{sign } c_x J(c_x c_z) e^{-c^2} d\vec{c}, \\ I_3 &= \int_{-\infty}^{\infty} c_x c_z J(c_x c_z) e^{-c^2} d\vec{c}, & I_4 &= \int_{-\infty}^{\infty} c_x c_z \text{sign } c_x J(c_x c_z \text{sign } c_x) e^{-c^2} d\vec{c}. \end{aligned} \quad (9)$$

The remaining expressions of this type vanish. Having specified the law of interaction of the molecules, one can calculate the values of the integrals (9).

We shall take into account that at a large distance from the wall $a_0^+ \equiv a_0^-$, $a_1^+ \equiv a_1^-$. If a constant velocity gradient is prescribed in the gas volume, then $a_1 = \text{const}$ and equations (8) take the form:

$$\frac{\pi}{2} \frac{\partial a_0}{\partial x} = a_1 I_2, \quad \frac{\pi}{2} \frac{\partial a_0}{\partial x} = \frac{2}{\sqrt{\pi}} a_1 I_3. \quad (10)$$

Hence it follows immediately that

$$I_3 = \frac{\sqrt{\pi}}{2} I_2. \quad (11)$$

Further, according to (5), (10) can be written in the form

$$-\frac{\pi}{2} \frac{\partial}{\partial x} 2u(x) = -l \frac{\partial u}{\partial x} I_2, \quad (12)$$

i.e. $I_2 = -\pi/l$. Thus, for hard spheres

$$I_2 = -\frac{4}{5\sqrt{\pi}} \frac{\pi}{\lambda} \simeq -0.452 \frac{\pi}{\lambda}, \quad I_3 = -0.4 \frac{\pi^*}{\lambda}. \quad (13)$$

Let us introduce the notation:

$$A_1 = \frac{I_1 - \frac{1}{2}\sqrt{\pi} I_2}{\pi(1 - \frac{1}{4}\pi)}, \quad B_1 = \frac{I_2 - \frac{1}{2}\sqrt{\pi} I_1}{\pi(1 - \frac{1}{4}\pi)}, \quad A_2 = \frac{I_2 - \frac{1}{2}\sqrt{\pi} I_3}{\pi(1 - \frac{1}{4}\pi)},$$

$$B_2 = \frac{I_3 - \frac{1}{2}\sqrt{\pi} I_2}{\pi(1 - \frac{1}{4}\pi)}, \quad A_3 = \frac{\frac{1}{2}\sqrt{\pi} I_4}{\pi(1 - \frac{1}{4}\pi)}, \quad B_3 = \frac{I_4}{\pi(1 - \frac{1}{4}\pi)}.$$

According to (11) and (12), $A_2 = -1/l$, $B_2 \equiv 0$. Equations (8) take the form:

$$\frac{d}{dx} a_0^\pm = (a_0^+ - a_0^-) A_1 + (a_1^+ + a_1^-) A_2 \mp (a_1^+ - a_1^-) A_3,$$

$$\frac{d}{dx} a_1^\pm = \pm (a_0^+ - a_0^-) B_1 + (a_1^+ - a_1^-) B_3. \quad (14)$$

As usual, we seek the solution of this system of first-order linear differential equations in the form:

$$a_i^\pm = \sum_j b_{ij}^\pm e^{\alpha_j x}.$$

The secular equation leads to the following eigenvalues:

$$\alpha_{1,2}^2 = 0, \quad \alpha_{3,4}^2 = -4B_1 A_3. \quad (15)$$

Thus, the solution of system (14) has the form:

$$a_i^\pm = c_i^\pm + d_i^\pm x + b_i^\pm e^{\alpha x} + g_i^\pm e^{-\alpha x},$$

where i takes the values $0, 1$; $\alpha^2 = -4B_1A_3$. Substitution into (14) of the particular solutions corresponding to each of the values of α (15) gives:

$$c_i^+ = c_i^-,$$

$$d_0^+ = d_0^- = 2A_2c_1^+, \quad d_1^+ = d_1^- = 0, \quad b_0^+ = \frac{A}{B}b_0^-, \quad b_1^+ = \frac{D}{B}b_0^-, \quad b_1^- = -\frac{C}{B}b_0^-,$$

$$g_0^+ = \frac{B}{A}g_0^-, \quad g_1^+ = \frac{C}{A}g_0^-, \quad g_1^- = -\frac{D}{A}g_0^-,$$

where it is denoted that

$$A = \alpha\sqrt{\pi} - 4B_1, \quad B = -\alpha\sqrt{\pi} - 4B_1, \quad C = 2(\alpha + B_1\sqrt{\pi}), \quad D = 2(-\alpha + B_1\sqrt{\pi}).$$

The four constants c_i^+ , b_0^- , and g_0^- will be found using conditions (5). Neglecting terms with $e^{-\alpha d}$, we obtain:

$$c_0^+ + \frac{B}{A}g_0^- = 0, \quad c_0^+ - 2c_1^+ \frac{d}{l} = 2u_z(d), \quad c_1^+ + \frac{C}{A}g_0^- = 0, \quad c_1^+ = -l \frac{\partial u_z}{\partial x}(d).$$

Hence

$$\begin{aligned} c_0^+ &= -\frac{B}{C}l \frac{\partial u}{\partial x}(d), & c_1^+ &= -l \frac{\partial u}{\partial x}(d), & g_0^- &= \frac{A}{C}l \frac{\partial u}{\partial x}(d), \\ & \left(2\frac{d}{l} - \frac{B}{C}\right)l \frac{\partial u}{\partial x}(d) & &= 2u(d). \end{aligned} \quad (16)$$

* In work (3) the values $I_2 = -0.4343\pi/\lambda$, $I_3 = -0.4001\pi/\lambda$, i.e., condition (11) is not satisfied.

Thus, finally, we have:

$$\begin{aligned} a_0^+ &= l \frac{\partial u}{\partial x}(d) \left(2\frac{x}{l} + \frac{B}{C}e^{-\alpha x} - \frac{B}{C}\right), \\ a_0^- &= l \frac{\partial u}{\partial x}(d) \left(2\frac{x}{l} + \frac{A}{C}e^{-\alpha x} - \frac{B}{C}\right), \end{aligned} \quad (17)$$

$$a_1^+ = l \frac{\partial u}{\partial x}(d) (e^{-\alpha x} - 1), \quad a_1^- = l \frac{\partial u}{\partial x}(d) \left(-\frac{D}{C}e^{-\alpha x} - 1\right).$$

Let us find the velocity profile of the ordered motion of the gas near the wall:

$$q(x) = \frac{1}{n} \int_{-\infty}^{\infty} v_z f d\bar{v} = \frac{1}{8} \bar{v} [\sqrt{\pi} (a_0^+ + a_0^-) + (a_1^+ - a_1^-)], \quad (18)$$

where $\bar{v} = (8kT/\pi m)^{1/2}$. Substituting here the values of a_i^\pm from (17), we obtain

$$\begin{aligned} q(x) &= \frac{1}{8} \bar{v} l \frac{\partial u}{\partial x} \left[\sqrt{\pi} \left(4 \frac{x}{l} - 2 \frac{B}{C} + \frac{A+B}{C} e^{-\alpha x} \right) + \left(1 + \frac{D}{C} \right) e^{-\alpha x} \right] \\ &= \frac{2\eta}{\pi \rho \bar{v}} \frac{\partial q}{\partial x}(d) \left[\sqrt{\pi} \left(4 \frac{x}{l} - 2 \frac{B}{C} \right) + \left(\frac{A+B}{C} \sqrt{\pi} + 1 + \frac{D}{C} \right) e^{-\alpha x} \right]. \end{aligned}$$

We have used here the well-known relation (see, for example, (4)) between the viscosity coefficient η of the gas and λ . For $x \rightarrow d$,

$$q(d) = \frac{4}{\sqrt{\pi}} \frac{\eta}{\rho \bar{v}} \left(2 \frac{d}{l} - \frac{B}{C} \right) \frac{\partial q}{\partial x}(d),$$

which exactly coincides with (16).

The quantity

$$-\frac{4}{\sqrt{\pi}} \frac{B}{C} \frac{\eta}{\rho \bar{v}} \frac{\partial q}{\partial x}(d)$$

is the phenomenological slip velocity q_c . Substituting here the value of B/C , we find

$$q_c = 2 \frac{\eta}{\rho \bar{v}} \frac{\alpha \sqrt{\pi} + 4B_1}{\alpha \sqrt{\pi} + \pi B_1} \frac{\partial q}{\partial x}(d).$$

Let us note that Maxwell obtained for q_c the value (diffuse reflection)

$$q_c^M = 2 \frac{\eta}{\rho \bar{v}} \frac{\partial q}{\partial x}(d).$$

In his calculations Maxwell proceeded from the assumption that molecules with $c_x < 0$ at any distance from the wall carry, in the direction of the wall, the component of momentum parallel to it

$$p_{xz}^- = -\frac{1}{2} \eta \frac{\partial q}{\partial x}.$$

Let us calculate this quantity by means of the distribution function we have obtained:

$$p_{xz} = \int_{-\infty}^{\infty} m(v_z - q) v_x f d\vec{v}.$$

The calculation gives for p_{xz}^+ and p_{xz}^- the values

$$p_{xz}^+(x) = -\frac{1}{2}\eta \frac{\partial q}{\partial x} \left[1 - \frac{B_1(\pi - 2)}{\alpha\sqrt{\pi} + \pi B_1} e^{-\alpha x} \right],$$

$$p_{xz}^-(x) = -\frac{1}{2}\eta \frac{\partial q}{\partial x} \left[1 + \frac{B_1(\pi - 2)}{\alpha\sqrt{\pi} + \pi B_1} e^{-\alpha x} \right].$$

It is seen from this that Maxwell's assumption (which, incidentally, was followed by almost all investigators of this problem) is correct only at an infinite distance from the wall. Let us find the numerical values of the coefficients in the expressions we have obtained. Taking into account (13), as well as the values given in (3),

$$I_1 = -1.0059 \pi/\lambda, \quad I_4 = -1.6982 \pi/\lambda,$$

we obtain

$$\alpha \simeq 7.56 \frac{1}{\lambda},$$

$$\frac{\alpha\sqrt{\pi} + 4B_1}{\alpha\sqrt{\pi} + \pi B_1} = 1.09, \quad \frac{B_1(\pi - 2)}{\alpha\sqrt{\pi} + \pi B_1} = 0.117.$$

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Note: Figure translations are in progress. See original paper for figures.

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