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Abstract

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ON THE COMPUTATION OF MULTIDIMENSIONAL INTEGRALS

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Consider the quadrature formula

$$\int_K f(P)\,dP \simeq \frac{1}{N}\sum_{i=1}^{N} f(P_i), \tag{1}$$

where $P = (x_1, \dots, x_d)$; K is the unit d-dimensional cube ($0 \le x_s \le 1$; $s = 1, 2, \dots, d$); the nodes $P_1, \dots, P_N$ belong to K.

To characterize the "quality" of an arbitrary net $P_1, \dots, P_N$, in ($^{1-4}$) a quantity $\varphi_\infty(P_1, \dots, P_N)$ was introduced, which enters into the error estimate of formula (1) for functions $f(P)$ of the class H_α:

$$\left|\int_K f(P)\,dP - \frac{1}{N}\sum_{i=1}^{N} f(P_i)\right| = O\left(\frac{\varphi_\infty(P_1, \dots, P_N)\ln^d N}{N^\alpha}\right)$$

(the class H_α (^{3}) is the multidimensional analogue of the class $\mathrm{Lip}\,\alpha$; $0 < \alpha \le 1$). In the present article we give examples of good nets (^{3}) and (^{4}), as well as an infinite sequence (^{5}), any initial segment of which is a good net.

From the estimates obtained for φ_∞ it follows that, when any of these nets is used, the order of convergence of formula (1) for functions of the class H_α is $O(1/N^{\alpha-\varepsilon})$ ($\varepsilon > 0$ arbitrarily small). It is also known (^{4}) that the order of convergence of quadrature formulas for the class H_α cannot be better than $1/N^\alpha$ (even if formulas with weights are allowed).

It is interesting to compare this result with the estimate for Monte Carlo methods. The Monte Carlo method, using formula (1) with random nodes $P_1, \dots, P_N$, with high probability guarantees the order of convergence $1/\sqrt{N}$ for any function with integrable square. Consequently, for the classes H_α with $\alpha > 1/2$ there exist quadrature formulas that ensure a better order of convergence than the Monte Carlo method, while for broader classes including $H_{1/2}$ there are no such formulas.

Definition of $\varphi_\infty(P_1, \dots, P_N)$. Let Π be an arbitrary dyadic parallelepiped*, belonging to K. If the origin of coordinates is moved to the center of Π, then the coordinate planes divide this parallelepiped into 2^d parts– "octants" for $d = 3$. Denote by V^+ (respectively V^-) the union of all positive (negative) "octants." Obviously, the volumes $|V^+| = |V^-| = 0.5|\Pi|$. The nonuniformity of the arrangement of the net $P_1, \dots, P_N$ with respect to Π can be characterized by the quantity $|S_N(V^+) - S_N(V^-)|$, where $S_N(V)$ is the number of points of the net belonging to V. The largest of such nonuniformities (over all Π) will be called the d-dimensional nonuniformity of the net.

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* By dyadic intervals we shall mean all intervals obtained by dividing $[0, 1]$ into 2^{m-1} equal parts ($m = 1, 2, \dots$); intervals are considered closed on the left and open on the right if the right endpoint is different from 1. Dyadic parallelepipeds are products of dyadic intervals.

Further, consider all s-dimensional grids consisting of projections of the points $P_1, \dots, P_N$ onto s-dimensional coordinate faces of the cube K ($s = 1, 2, \dots, d-1$), and find their s-dimensional irregularities. The value $\varphi_\infty(P_1, \dots, P_N)$ is equal to the greatest among all the irregularities of the grid and its projections.

Clearly, $\varphi_\infty(P_1, \dots, P_N)$ is an integer satisfying the inequalities

$$1 \leqslant \varphi_\infty(P_1, \dots, P_N) \leqslant N. \tag{2}$$

The upper bound (2) is sharp. The exact value of the lower bound for arbitrary d is unknown. However, if $d = 1$, then $\inf \varphi_\infty = 1$, and if $d = 2$, then $\inf \varphi_\infty = 2$ (for $N \geqslant 2$).

In order that an infinite sequence of points $P_1, P_2, \dots, P_N, \dots$ be uniformly distributed in K, it is necessary and sufficient that

$$\lim_{N \to \infty} \varphi_\infty(P_1, \dots, P_N)/N = 0.$$

Example 1. Fix a prime number $N > 3$ and consider the grid consisting of N points P_i ($i = 1, 2, \dots, N$) with coordinates

$$P_i = \left(\left\{ \frac{a_1}{N} i \right\}, \left\{ \frac{a_2}{N} i \right\}, \dots, \left\{ \frac{a_d}{N} i \right\} \right); \tag{3}$$

here $a_1, \dots, a_d$ are integers, $1 \leqslant a_s \leqslant N - 1$; $\{z\}$ is the fractional part of the number z.

In ([5]) the concept of optimal coefficients $a_1, \dots, a_d$ was introduced, for the choice of which the grids (3) ensure almost the best order of convergence of formula (1) for certain classes of differentiable periodic functions. Estimates for a broader class of periodic functions are given in ([6]).

For the grid (3) with optimal $a_1, \dots, a_d$,

$$\varphi_\infty(P_1, \dots, P_N) \leqslant B_1 \ln^d N,$$

where $B_1 \to 2d(2 + 8/\pi)^d$ as $N \to \infty$.

Definition of the sequences $p_r(i)$. Fix a natural number $r \geqslant 2$. The following three definitions of the infinite sequence $p_r(i)$, $i = 1, 2, \dots$, are equivalent.

1) If in the r-ary system

$$i = a_m a_{m-1} \dots a_1 a_0,$$

then (also in the r-ary system)

$$p_r(i) = 0, a_0 a_1 \dots a_{m-1} a_m {}^*.$$

2) Recursive definition. In the r-ary fraction

$$p_r(i) = 0, a_0 a_1 \dots a_m 00 \dots$$

find the leading digit different from $r-1$. Let it be a_k. To obtain $p_r(i+1)$ one must replace a_k by $1+a_k$, and replace all digits preceding a_k (if $k > 0$) by zeros; the following digits $a_{k+1}, a_{k+2}, \dots$ remain unchanged**.

3) Recursive definition by groups. 1°. $p_r(r^k) = r^{-k-1}$; $k = 0, 1, 2, \dots$. 2°. $p_r(r^k + j) = p_r(r^k) + p_r(j)$ for $j = 1, 2, \dots, r^{k+1} - r^k - 1$.

The sequence $p_2(i)$, apparently, was first indicated by van der Corput (see ([7])) and independently of him in ([1]). The generalization to arbitrary r belongs to Hammersley (see ([8])). Each of these sequences is uniformly distributed on $(0, 1)$. In particular, $\varphi_\infty(p_2(1), \dots, p_2(N)) = 1$ for any N.

Example 2. Choose pairwise relatively prime natural numbers $r_1, \dots, r_{d-1}$, and let $N > \max r_s$. Consider the grid consisting of N points P_i

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* Expanded notation: if $i = a_m r^m + \dots + a_1 r + a_0$, then

$$p_r(i) = a_0 r^{-1} + a_1 r^{-2} + \dots + a_{mr}^{-m-1},$$

each a_s may take the values $0, 1, 2, \dots, r-1$.

** Expanded notation:

$$p_r(i+1) = p_r(i) + r^{-k} + r^{-k-1} - 1.$$

$(i = 1, 2, \dots, N)$ with coordinates

$$P_i = (i/N,\ p_{r_1}(i),\ p_{r_2}(i), \dots,\ p_{r_{d-1}}(i)). \tag{4}$$

In paper (8), for the nets (4) and (5), estimates were obtained for the difference $S_N(\Pi) - N|\Pi|$ (for parallelepipeds of a special form). Using this result, it is not difficult to compute that

$$\varphi_\infty(P_1, \dots, P_N) \leq B_2 \ln^{d-1} N,$$

where

$$B_2 = 4^d \prod_{s=1}^{d-1} (3r_s - 2)/\ln r_s.$$

Example 3. Consider the infinite sequence of points P $(i = 1, 2, \dots)$ with coordinates

$$P_i = (p_{r_1}(i),\ p_{r_2}(i), \dots,\ p_{r_d}(i)), \tag{5}$$

where $r_1, \dots, r_d$ are natural pairwise coprime numbers. As in the preceding example, we obtain that for any $N > \max r_s$

$$\varphi_\infty(P_1, \dots, P_N) \leq B_3 \ln^d N,$$

where

$$B_3 = 4^d \prod_{s=1}^{d} (3r_s - 2)/\ln r_s.$$

In many problems solved by the Monte Carlo method, mathematical expectations of certain functions are computed, i.e., definite integrals. In such calculations it is sometimes expedient, instead of random points uniformly distributed in K, to use points of the sequence (5): if the integrand is "sufficiently good" (for example, belongs to H_α with $\alpha > 1/2$), then the order of convergence of formula (1) will prove better than the order of convergence of the Monte Carlo methods. In general, convergence is guaranteed for any bounded and Riemann-integrable function. Unfortunately, it apparently is not possible to use the variance as a measure of the error in this method of computation.

Remark. The error estimate for formula (1) written at the beginning of the article is not sharp: it is obtained from sharp estimates for the classes S_p by means of the embedding theorem (3). However, for good nets (Examples 1, 2, and 3) it makes it possible to determine the order of convergence of formula (1) with accuracy up to an arbitrarily small $\varepsilon > 0$. For bad nets this estimate becomes less effective. Consider, for example, the uniform (cubic) net consisting of $N = n^d$ points $(n \geq 2)$ with coordinates $(i_1/n, i_2/n, \dots, i_d/n)$, where $i_s =$

$0, 1, 2, \ldots, n-1$. For such a net the largest one-dimensional nonuniformities are equal to n^{d-1}; hence $\varphi_\infty = N^{1-1/d}$. The error estimate for the classes H_α was obtained in (2). Its order is $N^{-\alpha/d}$, whereas from the estimate under consideration there follows only $O(N^{1+\varepsilon-\alpha-1/d})$.

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Note: Figure translations are in progress. See original paper for figures.