On Tests for Convergence of Multiple Series with Positive Terms

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Abstract

Full Text

On Tests for Convergence of Multiple Series with Positive Terms

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In my paper ([1]) the following was given.

Test I. If for every system of nonnegative constants

$$c_1, c_2, \dots, c_n, \tag{1}$$

where

$$\sum_{i=1}^{n} c_i > 0,$$

the inequality

$$\overline{\lim}_{t\to\infty} f^{1/t}(c_1 t, c_2 t, \dots, c_{nt}) = q(c_1, c_2, \dots, c_n) < 1, \tag{2}$$

holds, then the series

$$\sum_{m_1, m_2, \dots, m_n = 0}^{\infty} f(m_1, m_2, \dots, m_n) \tag{3}$$

converges. If, however, for at least one system (1) the inequality

$$q(c_1, c_2, \dots, c_n) > 1$$

holds, then the series (3) diverges.

Taking into account the fact that, if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ exists, then it is equal to $\lim_{k\to\infty} \sqrt[k]{a_k}$, and using the preceding test, one can obtain the following:

Test II. If for every system of constants (1) there exists the limit

$$\lim_{k\to\infty} \frac{f[c_1(k+1), c_2(k+1), \dots, c_n(k+1)]}{f(c_1 k, c_2 k, \dots, c_{nk})} = q(c_1, c_2, \dots, c_n) < 1, \tag{4}$$

then the series (3) converges. If, however, for at least one system (1) the inequality

$$q(c_1, c_2, \dots, c_n) > 1, \tag{5}$$

holds, and the sequence

$$f(c_1 k, c_2 k, \dots, c_{nk}), \qquad k = 1, 2, \dots, \tag{6}$$

decreases monotonically, then the series diverges.

This test makes it possible to generalize the tests of V. Ermakov ([2]) and d' Alembert to the case of n-fold series.

The series

$$\sum_{m_1, m_2, \dots, n_n = 0}^{\infty} F(m_1, m_2, \dots, m_n), \tag{7}$$

where

$$F(m_1, m_2, \dots, m_n) = f[\delta_1(m_1), \delta_2(m_2), \dots, \delta_n(m_n)]\, \delta_1'(m_1) \cdots \delta_n'(m_n), \tag{8}$$

is called **condensed** with respect to series (3).

Let the function $\varphi(t)$:

$$\left.\begin{array}{ll} 1) & \text{be positive;} \\ 2) & \text{be continuously differentiable;} \\ 3) & \text{increase monotonically, } \lim_{t\to\infty} \varphi(t) = \infty, \end{array}\right\} \tag{9}$$

and $f(x_1, x_2, \dots, x_n)$:

$$\left.\begin{array}{ll} 1) & \text{be positive;} \\ 2) & \text{decrease monotonically along any ray issuing from the origin;} \\ 3) & \text{be continuous outside a certain } n\text{-dimensional sphere with center at the origin.} \end{array}\right\} \tag{10}$$

Then the following holds:

Theorem. If the functions $\delta_i(t)$ $(i = 1, 2, \dots, n)$ satisfy conditions (9), and the functions $f(x_1, x_2, \dots, x_n)$ and $F(x_1, x_2, \dots, x_n)$ satisfy conditions (10), then series (3) and the condensed series (7) are associated, i.e., from the convergence of either one of them follows the convergence of the other.

The proof is carried out by applying Cauchy' s integral test ([3]) to the series (3) and (7).

For single series this theorem was proved by N. V. Bugaev ([4]), but its formulation requires clarification, namely, it is required that the terms of the condensed series also decrease monotonically. V. A. Zmorovich ([6]) gave an example of a single series satisfying the conditions of Bugaev' s theorem, but contradicting its conclusion.

To derive Ermakov' s test, take a system of condensers $\{\delta_i(t)\}$ $(i = 1, 2, \dots, n)$, satisfying conditions (9), construct the condensed series for series (3), and apply convergence test II. We have

$$q = \lim_{k\to\infty} \frac{f\{\delta_1[c_1(k+1)], \delta_2[c_2(k+1)], \dots, \delta_n[c_n(k+1)]\}\delta_1'[c_1(k+1)]\cdots\delta_n'[c_n(k+1)]}{f[\delta_1(c_1k), \delta_2(c_2k), \dots, \delta_n(c_{nk})]}. \tag{11}$$

Introduce functions $\varphi_i(x_i)$, satisfying the conditions

$$\delta_i[c_i(k+1)] = \varphi_i[\delta_i(c_{ik})] = \varphi_i(x_i), \tag{12}$$

where $x_i = c_{ik}$ $(i = 1, 2, \dots, n)$.

Differentiating (12) and substituting into (11), we obtain

$$q = \lim_{x_1, x_2, \dots, x_n\to\infty} \frac{f[\varphi_1(x_1), \varphi_2(x_2), \dots, \varphi_n(x_n)]\varphi_1'(x_1)\varphi_2'(x_2)\cdots\varphi_n'(x_n)}{f(x_1, x_2, \dots, x_n)}. \tag{13}$$

Hence it follows:

Test III. If the function $f(x_1, x_2, \dots, x_n)$ and the corresponding condensed function $F(x_1, x_2, \dots, x_n)$ satisfy conditions (10), and the condensers $\varphi_i(t)$ satisfy conditions (9), then series (3) converges for $q < 1$ and diverges for $q > 1$, where q is determined from (13).

Putting $\varphi(t) = e^t$, we have the following:

Test IV. If the functions

$$f(x_1, x_2, \dots, x_n) \quad \text{and} \quad f(e^{x_1}, e^{x_2}, \dots, e^{x_n})\, e^{x_1+x_2+\cdots+x_n}$$

satisfy conditions (10) and

$$q = \lim_{x_1, x_2, \dots, x_n\to\infty} \frac{f(e^{x_1}, e^{x_2}, \dots, e^{x_n})e^{x_1+x_2+\cdots+x_n}}{f(x_1, x_2, \dots, x_n)},$$

then series (3) converges for $q < 1$ and diverges for $q > 1$.

Putting $\varphi(t) = t + 1$, we obtain:

Test V. If the function $f(x_1, x_2, \dots, x_n)$ satisfies conditions (10) and

$$q = \lim \frac{f(x_1 + 1, x_2 + 1, \dots, x_n + 1)}{f(x_1, x_2, \dots, x_n)},$$

then series (3) converges for $q < 1$, and diverges for $q > 1$.

For simple series, Tests III and IV were obtained by Ermakov, but in his formulation the test is not always correct—he has no monotonicity condition for the terms of the condensed series. For a simple series A. Ostrovskiĭ [5] refined the formulation of Ermakov’ s test.

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References

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Note: Figure translations are in progress. See original paper for figures.