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ON FINITE INTEGRAL HANKEL TRANSFORMS

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Abstract

Full Text

MATHEMATICS

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ON FINITE INTEGRAL HANKEL TRANSFORMS

(Presented by Academician I. N. Vekua, 3 V 1961)

The known finite integral Fourier and Hankel transforms are introduced from expansions of a function $f(x)$ in Fourier, Fourier-Bessel, and Bessel-Dini series. It is natural, in order to obtain new finite integral transforms, to consider more general orthogonal series—series in eigenfunctions.

Let us expand the function $f(x)$ in a Fourier series

$$f(x) = \sum_n \frac{\int_a^b x Q_n(\lambda, x) f(x) dx}{\int_a^b x Q_n^2(\lambda, x) dx} Q_n(\lambda, x) \quad (1)$$

in the eigenfunctions of the Sturm-Liouville problem

$$(xu')' + \left[\lambda^2 x - \frac{m^2}{x} \right] u(x) = 0, \quad a \leq x \leq b, \quad h \geq 0, \quad H \geq 0;$$

$$u' - hu|_{x=a} = 0; \quad u' + Hu|_{x=b} = 0. \quad (2)$$

It is not difficult to see that the eigenfunctions are equal to*

$$Q_n(\lambda_n x) = \lambda_n V_m(\lambda_n x) + HW_m(\lambda_n x), \quad (3)$$

where the notation has been introduced

$$V_m(\lambda_n x) = Y'_m(\lambda_n b) J_m(\lambda_n x) - J'_m(\lambda_n b) Y_m(\lambda_n x),$$

$$W_m(\lambda_n x) = Y_m(\lambda_n b) J_m(\lambda_n x) - J_m(\lambda_n b) Y_m(\lambda_n x). \quad (4)$$

The eigenvalues λ_n are the positive roots of the transcendental equation

$$\lambda_n^2 V_m'(\lambda_n a) + H\lambda_n W_m'(\lambda_n a) - h\lambda_n V_m(\lambda_n a) - hHW_m(\lambda_n a) = 0. \quad (5)$$

Obviously (*),

$$\int_a^b x Q_m^2(\lambda_n x) dx = \frac{x^2}{2} \left\{ \left(1 - \frac{m^2}{\lambda_n^2 x^2} \right) Q_m^2(\lambda_n x) + Q_{m'}^2(\lambda_n x) \right\} \Big|_a^b,$$

$$f(x) = 2 \sum_n \frac{\lambda_n^2 \int_a^b x [\lambda_n V_m(\lambda_n x) + HW_m(\lambda_n x)] f(x) dx}{[b^2 \lambda_n^2 - m^2 + b^2 H^2] \lambda_n^2 V_m^2(\lambda_n b) - [a^2 \lambda_n^2 - m^2 + a^2 h^2] Q_m^2(\lambda_n a)} Q_m(\lambda_n x). \quad (6)$$

* The prime everywhere denotes differentiation with respect to $\lambda_n x$.

Summation in (6) is carried out over all positive roots of equation (5). But

$$V_m(\lambda_n b) = \frac{2}{\pi \lambda_n b}; \quad Q_m(\lambda_n a) = \frac{2}{\pi a} \frac{\lambda_n J_m'(\lambda_n b) + H J_m(\lambda_n b)}{\lambda_n J_m'(\lambda_n a) - h J_m(\lambda_n a)},$$

and the series (6) can be represented in the form

$$f(x) = \frac{\pi^2}{2} \sum_n \times$$

$$\times \frac{\lambda_n^2 [\lambda_n J_m'(\lambda_n a) - h J_m(\lambda_n a)]^2 \int_a^b x [\lambda_n V_m(\lambda_n x) + HW_m(\lambda_n x)] f(x) dx}{\frac{b^2 \lambda_n^2 - m^2 + b^2 H^2}{b^2} [\lambda_n J_m'(\lambda_n a) - h J_m(\lambda_n a)]^2 - \frac{a^2 \lambda_n^2 - m^2 + a^2 h^2}{a^2} [\lambda_n J_m'(\lambda_n b) + H J_m(\lambda_n b)]^2} \times$$

$$\times Q_m(\lambda_n x). \quad (7)$$

Introduce the finite integral transform

$$\bar{f}(\lambda_n) = \int_a^b x [\lambda_n V_m(\lambda_n x) + HW_m(\lambda_n x)] f(x) dx. \quad (8)$$

The inversion formula for it is obtained from expansion (7). Following Sneddon, we shall call the transform (8)–(7) a Hankel transform. For particular values of h and H we obtain the known finite Hankel transforms.

Let us pass to the interval $[0, b]$. For fixed λ , every solution of Bessel's equation has a finite number of zeros on the interval $[a, b]$ as $a \rightarrow 0$. It follows from this ⁽²⁾ that the equation $(xu')' + [\lambda^2 x - \frac{m^2}{x}]u(x) = 0$ and the boundary condition $u' + Hu|_{x=b} = 0$ determine a discrete spectrum of eigenvalues on $[0, b]$; the corresponding eigenfunctions are square-integrable. For $m \geq \frac{1}{2}$, only $J_m(\lambda_n x)$ is square-integrable in a neighborhood of zero, and the requirement of integrability replaces the boundary condition $u' - hu|_{x=a} = 0$. The eigenfunctions in this case are $J_m(\lambda_n x)$, and the eigenvalues satisfy the equation $\lambda_n J'_m(\lambda_n b) + HJ_m(\lambda_n b) = 0$. This case corresponds to the second Hankel transform, and for $H \rightarrow \infty$ to the first.

If $m < \frac{1}{2}$, $J_m(\lambda_n x)$ and $Y_m(\lambda_n x)$ are square-integrable, and the requirement of integrability does not replace the boundary condition at the left endpoint. Let us see how the latter is transformed as $a \rightarrow 0$.

Expression (3) for the eigenfunctions can be rewritten in the form

$$Q_m(\lambda_n x) = -\frac{1}{\sin \pi m} \{ [\lambda_n J'_{-m}(\lambda_n b) + HJ_{-m}(\lambda_n b)] J_m(\lambda_n x) - [\lambda_n J'_m(\lambda_n b) + HJ_m(\lambda_n b)] J_{-m}(\lambda_n x) \}.$$

For fixed λ , $m \neq 0$, and $x \rightarrow 0$, the estimates

$$J_m(\lambda x) = \frac{(\lambda x)^m}{2^m \Gamma(1+m)} + O(x^{m+2}); \quad J'_m(\lambda x) = m \frac{(\lambda x)^{m-1}}{2^m \Gamma(1+m)} + O(x^{m+1}).$$

are valid. Therefore the characteristic equation, as $a \rightarrow 0$, takes the form

$$\begin{aligned} & \frac{\lambda_n^m}{2^m \Gamma(1+m)} [\lambda_n J'_{-m}(\lambda_n b) + HJ_{-m}(\lambda_n b)] (ma^{m-1} - ha^m) + \\ & + \frac{\lambda_n^{-m}}{2^{-m} \Gamma(1-m)} (ma^{-m-1} + ha^{-m}) [\lambda_n J'_m(\lambda_n b) + \\ & + HJ_m(\lambda_n b)] + O(a^{-m+1}) + O(a^{-m+2}) = 0. \end{aligned} \quad (9)$$

We shall vary h together with a according to the law

$$h = \frac{C \cdot 2^{-m} \Gamma(1-m) ma^{m-1} - 2^m \Gamma(1+m) ma^{-m-1}}{C \cdot 2^{-m} \Gamma(1-m) a^m + 2^m \Gamma(1+m) a^{-m}}$$

(C is an arbitrary constant), which is possible, since the spectrum is completely determined by the boundary condition at the right end.

Since the order of the expression $ma^{m-1} - ha^m$ with respect to a is $m - 1$, in equation (9) one may omit the terms of orders $-m + 1$ and $-m + 2$, if $-m + 1 > m - 1$, and for $m < 1/2$ the eigenvalues satisfy the equation

$$\lambda_n^{2m} [\lambda_n J'_{-m}(\lambda_n b) + H J_{-m}(\lambda_n b)] + C [\lambda_n J'_m(\lambda_n b) + H J_m(\lambda_n b)] = 0. \quad (10)$$

The eigenfunctions are

$$Q_m(\lambda_n x) = C J_m(\lambda_n x) + \lambda_n^{2m} J_{-m}(\lambda_n x). \quad (11)$$

Thus, for $m < 1/2$ one may introduce the Hankel transform

$$\bar{f}(\lambda_n) = \int_0^b x [C J_m(\lambda_n x) + \lambda_n^{2m} J_{-m}(\lambda_n x)] f(x) dx,$$

where C is an arbitrary constant, and λ_n are the positive roots of the transcendental equation (10). This transform generalizes the second Hankel transform, obtained for an infinitely large value of C .

Repeating analogous arguments for $m = 0$ and using the estimates

$$Y_0(\lambda x) = \frac{2}{\pi} \left[\gamma + \ln \frac{\lambda}{2} \right] [1 + O(x^2)]; \quad Y'_0(\lambda x) = \frac{2}{\pi \lambda x} + O(x |\ln x|),$$

we obtain an integral transform whose kernel is equal to

$$x \left[\left(\frac{2}{\pi} \ln \lambda_n - C \right) J_0(\lambda_n x) - Y_0(\lambda_n x) \right], \quad (12)$$

and the eigenvalues satisfy the equation

$$[\lambda_n Y'_0(\lambda_n b) + H Y_0(\lambda_n b)] - [\lambda_n J'_0(\lambda_n b) + H J_0(\lambda_n b)] \left(\frac{2}{\pi} \ln \lambda_n - C \right) = 0. \quad (13)$$

Let us show that the finite integral transform introduced in (8)–(7) makes it possible to solve effectively boundary-value problems of the third kind for equations for which the corresponding homogeneous equations admit at least a partial separation of variables and represent, with respect to the separable variable, a Bessel equation of order m for hollow cylindrical bodies.

Suppose it is required to integrate the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - \frac{m^2}{r^2} u + D_1(z, \theta, t) u = f(r, z, \theta, t) \quad (14)$$

under boundary conditions with respect to r

$$\frac{\partial u}{\partial r} - hu \Big|_{r=r_0} = F_1(z, \theta, t); \quad \frac{\partial u}{\partial r} + Hu \Big|_{r=R} = F_2(z, \theta, t); \quad h \geq 0; \quad H \geq 0. \quad (15)$$

We pass to transforms by multiplying both sides of equation (14) by $rQ_m(\lambda_n r)$ and integrating with respect to r from r_0 to R , where $Q_m(\lambda_n r)$ are the eigenfunctions of problem (2).

We then obtain the equation

$$D_1 \bar{u}(\lambda_n) - \lambda_n^2 \bar{u}(\lambda_n) = r_0 Q_m(\lambda_n r_0) F_1(z, \theta, t) - R Q_m(\lambda_n R) F_2(z, \theta, t) + \int_{r_0}^R r Q_m(\lambda_n r) f dr, \quad (16)$$

which does not contain the variable r and includes the boundary conditions with respect to r .

The desired solution is represented by the series in eigenfunctions

$$u = \frac{\pi^2}{2} \sum_n \left\{ \left[\lambda_n^2 [\lambda_n J'_m(\lambda_n r_0) - h J_m(\lambda_n r_0)]^2 \int_{r_0}^R r [\lambda_n V'_m(\lambda_n r) + H W'_m(\lambda_n r)] u dr \right] \left[\frac{R^2 \lambda_n^2 - m^2 + R^2 H^2}{R^2} [\lambda_n J_m(\lambda_n R) - H J_m(\lambda_n R)] \right] \right\} \quad (17)$$

The summation in (17) extends over all positive roots of the transcendental equation (5).

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REFERENCES

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- ² B. M. Levitan, *Expansion in Eigenfunctions*, 1950.

Note: Figure translations are in progress. See original paper for figures.

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