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# Mathematics

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**Abstract**

**Full Text**

*Mathematics*

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## ON THE QUESTION OF THE REGULARITY OF A CONVEX SURFACE WITH A REGULAR METRIC IN EUCLIDEAN SPACE

The problem of the regularity of a convex surface with a regular metric in Euclidean space was essentially solved in the author's paper <sup>(1)</sup>. Namely, the following theorem was proved.

*If a convex surface has a regular ( $n$  times differentiable,  $n \geq 5$ ) metric and positive Gaussian curvature, then the surface itself is regular (at least  $n - 1$  times differentiable). If, moreover, the metric of the surface is analytic, then the surface is analytic.*

We say "essentially" solved, since it remained unclear what happens in the cases  $n = 3$  and  $n = 4$ . For  $n = 2$  the answer is given by a theorem of A. D. Aleksandrov <sup>(2)</sup>, according to which a convex surface with bounded positive specific curvature, in particular with a twice differentiable metric and positive Gaussian curvature, is smooth, i.e. at least once differentiable. At present, thanks to the works of Nirenberg <sup>(3)</sup> and Heinz <sup>(4)</sup>, the regularity theorem can also be proved in the cases  $n = 3$  and  $n = 4$ . We shall show this.

The proof of the theorem on the regularity of a convex surface with a regular metric contained in <sup>(1)</sup> consists, in broad outline, of the following. Let  $\Phi$  be a convex surface with a regular metric and positive curvature, and let  $P$  be a point on this surface. To prove the regularity of the surface  $\Phi$  in a neighborhood of  $P$ , introduce such a parametrization  $u, v$  in which the metric of the surface has the regularity specified in the hypothesis of the theorem. By a plane parallel to the tangent plane at the point  $P$ , cut off from the surface  $\Phi$  a small cap  $\omega$ , and denote by  $G_\omega$  the domain in the coordinate plane  $u, v$  corresponding to the cap.

Construct a sequence of closed analytic curves  $\gamma_k$  inside the domain  $G_\omega$ , converging to the boundary of the domain, such that the geodesic curvature of each curve  $\gamma_k$  in the metric of the cap, given by the line element

$$ds^2 = E du^2 + 2F du dv + G dv^2$$

is strictly positive.

Next construct a sequence of analytic metrics

$$ds_n^2 = E_n du^2 + 2F_n du dv + G_n dv^2$$

in the domain  $G_\omega$ , converging to the metric  $ds^2$  of the cap  $\omega$ , in the class  $C^n$ , i.e. such that the coefficients  $E_k, F_k, G_k$  of the line element  $ds_k^2$  converge to the coefficients  $E, F, G$  of the line element  $ds^2$  together with derivatives up to order  $n$ . Without loss of generality one may assume that the geodesic curvature of the curve  $\gamma_k$  in the metric  $ds_k^2$  is strictly positive.

The metric  $ds_k^2$ , considered in the domain  $G_{\omega_k}$  bounded by the curve  $\gamma_n$ , can be realized by an analytic cap  $\omega_k$  <sup>(1)</sup>. As  $k \rightarrow \infty$ , the sequence ...

the sequence of caps  $\omega_k$  converges to the cap  $\bar{\omega}$ , equal to  $\omega$  <sup>(1)</sup>. Thus, in order to prove the regularity of  $\omega$  in a neighborhood of  $P$ , it suffices to prove the regularity of  $\bar{\omega}$ .

Heinz, in paper <sup>(4)</sup>, proved the following theorem.

*Let  $r(u, v)$  be a twice differentiable vector function, defined in a domain  $\Omega$  of the  $u, v$ -plane;*

$$ds^2 = E du^2 + 2F du dv + G dv^2$$

*is the linear element of the surface  $\Phi : r = r(u, v)$ . Suppose that the coefficients of the form  $ds^2$  are bounded together with their derivatives up to the third order and with the reciprocal of the discriminant of the form by a constant  $a$ , that the Gaussian curvature of the surface  $\Phi$  is everywhere greater than  $b > 0$ , and that the integral mean curvature is bounded by a constant  $c < \infty$ . Then, on the set of interior points of  $\Omega$  whose distance from its boundary is not less than  $\rho > 0$ , one can give an estimate for the moduli of the second derivatives of  $r(u, v)$  depending only on the quantities  $a, b, c, \rho$ . Moreover, whatever positive number  $\nu < 1$  may be chosen, for the Hölder constants of the second derivatives of  $r$  with exponent  $\nu$  an estimate can be given in terms of the same quantities and the number  $\nu$ .*

Applying this theorem to the sequence of caps  $\omega_k$  and passing to the limit as  $k \rightarrow \infty$ , we conclude that the limiting cap  $\bar{\omega}$  must belong to the class  $C^{2+\nu}$ ,  $0 < \nu < 1$ , i.e., the vector function  $r(u, v)$  defining it has second derivatives satisfying a Hölder condition with any exponent  $\nu$ ,  $0 < \nu < 1$ .

The further regularity of the limiting cap  $\bar{\omega}$  is obtained by means of Nirenberg's theorem on the character of the regularity of a twice differentiable solution of an elliptic equation with regular coefficients, applying it to Darboux's equation. According to this theorem, from the twice differentiability of the surface  $\bar{\omega}$  and the  $n$ -fold differentiability of its metric it follows that the surface  $\bar{\omega}$  is differentiable at least  $n - 1$  times, and the  $(n - 1)$ -st derivatives of the function  $r(u, v)$  defining the surface satisfy a Hölder condition with any exponent  $\lambda$ ,  $0 < \lambda < 1$ .

Thus, if  $n \geq 3$ , then a convex surface with a metric of class  $C^n$  and positive Gaussian curvature belongs at least to the class  $C^{n-1+\alpha}$  for any  $\alpha$ ,  $0 < \alpha < 1$ .

An analogous result also holds for  $n = 2$ . It is obtained from another theorem of Heinz<sup>(5)</sup>. Heinz proved that if a twice differentiable function  $z(x, y)$  in a domain  $\Omega$  of the  $xy$ -plane satisfies the condition

$$0 < \alpha < z_{xx}z_{yy} - z_{xy}^2 < \beta < \infty,$$

then on the set of interior points of  $\Omega$  whose distance from its boundary is not less than  $d > 0$ , the first derivatives of the function  $z(x, y)$  satisfy a Hölder condition with exponent  $\nu = \sqrt{\beta/\alpha}$  and with a constant depending only on  $\alpha, \beta, d$ , and

$$\delta = d \max_{\Omega} (z_x^2 + z_y^2)^{1/2}.$$

Applying this theorem to the caps  $\omega_k$  converging to  $\bar{\omega}$ , we conclude that in a neighborhood of the point  $P$  the first derivatives of the functions  $z_k(x, y)$  defining these surfaces uniformly satisfy a Hölder condition with any exponent  $\nu < 1$ . (For a small neighborhood,  $\beta/\alpha$  is arbitrarily close to one, in view of the continuity of the Gaussian curvature and the smoothness of the limiting surface.) Passing to the limit as  $k \rightarrow \infty$ , we conclude that the surface  $\omega = \bar{\omega}$  also has this property.

Thus, we arrive at a complete solution of the question of the regularity of a convex surface with a regular metric in the following form.

**Theorem.** If a convex surface has a regular metric of class  $C^n$  ( $n \geq 2$ ) and positive Gaussian curvature, then the surface itself belongs at least to the class  $C^{n-1+\alpha}$  for any  $\alpha$ ,  $0 < \alpha < 1$ . If the metric of the surface is analytic, then the surface is analytic.

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- <sup>5</sup> E. Heinz, Math. Zs., 72, No. 2, 107 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

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