ON A GENERALIZATION OF THE CONCEPT OF A BASIS AND OF N. K. BARI’ S THEOREMS ON BASES

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Abstract

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ON A GENERALIZATION OF THE CONCEPT OF A BASIS AND OF N. K. BARI' S THEOREMS ON BASES

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In the present paper the concept of a continual basis in a Hilbert space is introduced and its properties are studied. It seems to us that the results obtained are a natural generalization of the results of N. K. Bari ([1,2]), concerning discrete bases, and of the results of M. M. Dzhrbashyan and R. M. Martirosyan ([3]), concerning properties of biorthogonal kernels.

As is known, a sequence of elements $\{g_n\}_{n=1}^{\infty}$ of a Hilbert space $\mathfrak{H}$ is called its basis if for every $x \in \mathfrak{H}$ there exists a unique numerical sequence $\{\xi_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \xi_n g_n$.

If for every $x \in \mathfrak{H}$ the corresponding series $\sum_1^{\infty} \xi_n g_n$ converges weakly to x, then it also converges strongly (see ([4]), p. 310). Therefore a basis $\{g_n\}_{n=1}^{\infty}$ may be identified with a linear mapping G of the space $\mathfrak{H}$ into the space of numerical sequences: $(Gy)_n = (y, g_n)$, possessing the property that for every $x \in \mathfrak{H}$ there exists a unique sequence $\{\xi_n\}_{n=1}^{\infty}$ such that

$$(x, y) = \sum_1^{\infty} \xi_n \overline{(G, y)_n}$$

for all $y \in \mathfrak{H}$.

Let us introduce the following notation: $\mathfrak{H}$ is a separable Hilbert space; σ is a nonnegative measure given on the Borel sets of the complex plane Λ (or, in general, Λ is a space with nonnegative measure σ); M_σ is the linear manifold of classes of σ-measurable and almost everywhere finite functions, equivalent with respect to the measure σ.

Definition 1. A linear mapping $G : \mathfrak{H} \to M_\sigma$ is called **basic** if for every element $x \in \mathfrak{H}$ there exists a unique element $\xi \in M_\sigma$ such that

$$(x, y) = \int_\Lambda \xi(\lambda) \overline{(Gy)(\lambda)} \, \sigma(d\lambda) \quad \text{for all } y \in \mathfrak{H}. \tag{1}$$

Here and everywhere below, the integral $\int_\Lambda \cdots \sigma(d\lambda)$ is understood as

$$\lim_{r\to\infty} \int_{\Lambda_r \cap \Delta} \cdots \sigma(d\lambda),$$

where $\Lambda_r = \{\lambda \in \Lambda : \ |\lambda| \le r\}$. This understanding of the integral also determines the manner of "ordering" the values of the mapping $(Gy)(\lambda)$. We note that in the case of a classical basis $\{g_n\}_{n=1}^\infty$ we deal not simply with a countable set of elements, but with a set ordered in a definite way, i.e. with a sequence.

Putting $(Fx)(\lambda) = \xi(\lambda)$, we obtain a linear mapping $F : \mathfrak{H} \to M_\sigma$, by means of which equality (1) can be given the form

$$(x, y) = \int_\Lambda (Fx)(\lambda)\, \overline{(Gy)(\lambda)}\, \sigma(d\lambda). \tag{1'}$$

We shall call the mapping F **conjugate** with respect to G.

Definition 2. The mapping G is called **integrally bounded** if there exists a σ-complete class Ω of Borel sets $\Delta \subset \Lambda$ such that

$$\left| \int_\Delta (Gy)(\lambda)\, \sigma(d\lambda) \right| \le K_\Delta \|y\|, \qquad K_\Delta < \infty,\ \Delta \in \Omega, \tag{2}$$

for all $y \in \mathfrak{H}$. Here the class Ω is called σ-complete if

$$\int_\Delta \xi(\lambda)\, \sigma(d\lambda) = 0$$

for all $\Delta \in \Omega$ only when $\xi = 0$*.

If G is an integrally bounded basis mapping, then the conjugate mapping F is a basis mapping.

In view of $(1')$, it suffices to prove that

$$\int_\Lambda \eta(\lambda)\, \overline{(Fx)(\lambda)}\, \sigma(d\lambda) = 0$$

for all $x \in \mathfrak{H}$ only when $\eta = 0$. Let Δ be an arbitrary fixed set in the class Ω. From inequality (2) we conclude that there exists an element $x(\Delta) \in \mathfrak{H}$ such that

$$(x(\Delta), y) = \int_\Delta \overline{(Gy)(\lambda)}\, \sigma(d\lambda), \qquad y \in \mathfrak{H}. \tag{3}$$

But the mapping G is a basis mapping; hence $Fx(\Delta) = \chi_\Delta$, where $\chi_\Delta \in M_\sigma$ is the class of functions equivalent to the characteristic function of the set Δ.

Now suppose that for some $\eta \in M_\sigma$

$$\int_\Lambda \eta(\lambda)\,\overline{(Fx)(\lambda)}\,\sigma(d\lambda) = 0$$

for every $x \in \mathfrak{H}$. Taking $x = x(\Delta)$, $\Delta \in \Omega$, we obtain

$$\int_\Delta \eta(\lambda)\,\sigma(d\lambda) = 0,$$

which is possible only for $\eta = 0$, as was required to prove.

Definition 3. A basis mapping G is called **orthogonal** if $F = G$, where F is the conjugate mapping.

Every orthogonal mapping E is an isometric mapping of $\mathfrak{H}$ onto L^2_σ, and, conversely, every isometric mapping E of the space $\mathfrak{H}$ onto L^2_σ is orthogonal.

From this the following corollary is easily obtained:

An orthogonal mapping is integrally bounded.

In what follows we shall consider only basis mappings G that are integrally bounded together with their conjugate mappings F.

Definition 4. The mapping G is called a **Bessel** mapping if $\mathfrak{R}(F) \subseteq L^2_\sigma$, i.e., for every $x \in \mathfrak{H}$, $Fx \in L^2_\sigma$ ($\mathfrak{R}(F)$ is the range of the mapping F).

Theorem 1. *In order that the mapping G be a Bessel mapping, it is necessary and sufficient that for every orthogonal mapping E there exist a linear bounded operator $S : \mathfrak{H} \to \mathfrak{H}$ such that $F = ES$. In this case S^{-1} exists and $\mathfrak{D}(S^{-1}) = \mathfrak{R}(S) = \mathfrak{H}$.*

Sufficiency is obvious, since $\mathfrak{R}(F) = \mathfrak{R}(ES) \subseteq \mathfrak{R}(E) = L^2_\sigma$.

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* Definitions 1 and 2 were proposed by V. E. Lyantse.

Necessity. Since $\mathfrak{R}(F) \subset L^2_\sigma$, putting

$$s(x, y) = \int_\Lambda (Fx)(\lambda)\overline{(Ey)(\lambda)}\,\sigma(d\lambda), \tag{4}$$

for fixed x we have

$$|s(x, y)| \le \|Fx\|_{L^2_\sigma} \|Ey\|_{L^2_\sigma} = \|Fx\|_{L^2_\sigma} \|y\|_{\mathfrak{H}}.$$

Consequently, $\overline{s(x, y)}$ is a linear bounded functional in $y \in \mathfrak{H}$: $s(x, y) = (Sx, y)$, where S is a linear operator defined on all of $\mathfrak{H}$.

Further,

$$\left|s(x, E^{-1}\chi_\Delta)\right| = \left|\int_\Delta (Fx)(\lambda)\,\sigma(d\lambda)\right| \le K_\Delta \|x\|.$$

Thus, linear combinations of elements of the form $E^{-1}\chi_\Delta$ belong to the domain of definition $\mathfrak{D}(S^*)$ of the operator S^*. Therefore $\mathfrak{D}(S^*)$ is dense in $\mathfrak{H}$, and since $\mathfrak{D}(S) = \mathfrak{H}$, it follows that the operator S is bounded. Since E is an orthogonal mapping, we have

$$s(x,y) = (Sx, y) = \int_\Lambda (ESx)(\lambda)\overline{(Ey)(\lambda)}\,\sigma(d\lambda).$$

Comparing this representation with (4), we arrive at the conclusion that $F = ES$.

Let us prove that the operator S is invertible and that $\mathfrak{D}(S^{-1}) = \mathfrak{R}(S) = \mathfrak{H}$. Indeed, if $Sx = 0$, then $Fx = 0$ and $x = 0$, since the mapping F is basic. Consequently, S^{-1} exists. Next let $(Sx, y) = 0$ for all x. We shall prove that $y = 0$. We have

$$0 = (Sx, y) = \int_\Lambda (Fx)(\lambda)\overline{(Ey)(\lambda)}\,\sigma(d\lambda).$$

Since the mapping F is basic, the last equality is possible only when $Ey = 0$, whence it follows that $y = 0$. The theorem is proved.

Definition 5. A mapping G is called **Hilbertian** if $\mathfrak{R}(F) \supset L^2_\sigma$: for every $\xi \in L^2_\sigma$ there exists an element $x \in \mathfrak{H}$ such that $Fx = \xi$.

Theorem 2. In order that the mapping G be Hilbertian, it is necessary and sufficient that for every orthogonal mapping E there exist a linear bounded operator $T : \mathfrak{H} \to \mathfrak{H}$ such that $FT = E$. In this case T^{-1} exists and $\mathfrak{D}(T^{-1}) = \mathfrak{R}(T) = \mathfrak{H}$.

Sufficiency is obvious, since

$$\mathfrak{R}(F) \supset \mathfrak{R}(FT) = \mathfrak{R}(E) = L^2_\sigma.$$

Necessity. Let x be an arbitrary fixed element of the space $\mathfrak{H}$. Since $Ex \in L^2_\sigma$ and $\mathfrak{R}(F) \supset L^2_\sigma$, the equation $Fy = Ex$ is solvable with respect to $y \in \mathfrak{H}$. It is easy to see that this solution is unique. Put $y = Tx$. Then T is a linear operator defined on the whole space $\mathfrak{H}$. Let us prove its boundedness. Let Δ be an arbitrary bounded σ-measurable set. Then

$$\left|(Tx, G^{-1}\chi_\Delta)\right| = \left|\int_\Delta (FTx)(\lambda)\,\sigma(d\lambda)\right| = \left|\int_\Delta (Ex)(\lambda)\,\sigma(d\lambda)\right| \le K_\Delta \|x\|,$$

since the mapping E is integrally bounded. From the inequality it follows that $\mathfrak{D}(T^*)$ is dense in $\mathfrak{H}$, and since $\mathfrak{D}(T) = \mathfrak{H}$, the operator T is bounded. The invertibility of the operator T is obvious. Let us prove that $\mathfrak{R}(T) = \mathfrak{H}$. Suppose for any $x \in \mathfrak{H}$ that $(Tx, y) = 0$. Then

$$0 = (Tx, y) = \int_\Lambda (FTx)(\lambda)\overline{(Gy)(\lambda)}\sigma(d\lambda) = \int_\Lambda (Ex)(\lambda)\overline{(Gy)(\lambda)}\,\sigma(d\lambda).$$

It follows that $Gy = 0$, hence also $y = 0$, as was required to prove.

The following facts are easily established:

Theorem 3. If the mapping G is Besselian (Hilbertian), then the adjoint mapping F is Hilbertian (Besselian).

Corollary 1. In order that the mapping G be Besselian, it is necessary and sufficient that there exist a bounded positive operator A such that $F = GA$.

Corollary 2. In order that the mapping G be Hilbertian, it is necessary and sufficient that there exist a bounded positive operator B such that $G = FB$.

Definition 6. A mapping G is called a **Fischer–Riesz mapping** if it is simultaneously Besselian and Hilbertian.

From Theorem 3 and its corollaries there follows immediately

Theorem 4. *In order that the mapping G be a Fischer–Riesz mapping, it is necessary and sufficient that there exist a bounded strictly positive operator C such that $F = GC$.*

Let us consider examples.

1°. Let $g_1, g_2, \ldots$ be a basis in $\mathfrak{H}$, and let σ be the measure concentrated at the points $\lambda = 1, 2, \ldots, n, \ldots$, with $\sigma(\{n\}) = 1$. Put $(Gy)(n) = (y, g_n)$. Then the mapping G is basis and integrally bounded. Indeed, the class Ω of sets each of which consists of only one point (with a natural affix) is σ-complete, and

$$\left| \int_{\Delta=\{n\}} (Gy)(\lambda)\, \sigma(d\lambda) \right| = |(y, g_n)| \le \|g_n\|\, \|y\|.$$

Clearly, the adjoint mapping F is generated by the adjoint basis $f_1, f_2, \ldots$.

In the example under consideration all properties of the mapping G coincide with the properties of the basis $g_1, g_2, \ldots$, studied by N. K. Bari ([1],[2]).

2°. In the paper ([3]) M. M. Dzhrbashian and R. M. Martirosian considered "integral" operators G mapping the space $L^2_{\sigma_1}(a_1, b_1)$ into $L^2_{\sigma_2}(a_2, b_2)$, defined by relations of the form

$$\int_{a_2}^{b_2} y(t) e_\xi(t)\, d\sigma_2(t) = \int_{a_1}^{b_1} x(t) \overline{G(t, \xi)}\, d\sigma_1(t), \qquad y = Gx,$$

where, for each fixed $\xi \in (a_1, b_1)$, the kernel $G(t, \xi) \in L^2_{\sigma_1}$; $e_\xi(t) = 1$, $t \in [0, \xi)$, and $e_\xi(t) = 0$, $t \notin [0, \xi)$, for $\xi > 0$; $e_\xi(t) = -1$, $t \in [\xi, 0)$, and $e_\xi(t) = 0$, $t \notin [\xi, 0)$, for $\xi < 0$.

Identify the space $\mathfrak{H}$ with $L^2_{\sigma_1}$, and let $\sigma = \sigma_2$. Then, as is not difficult to see, the mapping G generated by the kernel $G(t, \xi)$ is integrally bounded. It is also easy to show that if the kernel $G(t,\xi)$ is Besselian or Hilbertian (for the definitions see ([3])), then the mapping G is Besselian or Hilbertian, respectively.

3°. Let $\mathfrak{H} = L^2(-\infty,\infty)$, and let the measure σ be concentrated on the real axis and coincide there with Lebesgue measure. Define the mapping G by means of the equality

$$(Gx)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\varphi(\lambda)t} x(t)\, dt,$$

where $\varphi(\lambda)$ is a certain function.

Then

$$(Fx)(\lambda) = \frac{\varphi'(\lambda)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\varphi(\lambda)t} x(t)\, dt.$$

It is easy to arrive at the following conclusion: if $\varphi(\lambda)$ increases monotonically, with $\varphi(-\infty) = -\infty$, $\varphi(\infty) = \infty$, then the mapping G is Besselian when

$$\operatorname*{vrai\,max}_{-\infty<\lambda<\infty} \varphi'(\lambda) < \infty,$$

and Hilbertian when

$$\operatorname*{vrai\,min}_{-\infty<\lambda<\infty} \varphi'(\lambda) > 0.$$

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