EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATORS (THE NONPERIODIC CASE)

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Abstract

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MATHEMATICS

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EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATORS (THE NON-PERIODIC CASE)

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1°. Let $\bar{C}$ be the space of functions $f(x)$ continuous on the interval $[-1, 1]$, with norm

$$\|f(x)\| = \max_{-1\le x\le 1} |f(x)|;$$

let $k \geq 0$ be an integer, and let $f^{(k)}(x)$ be the k-th derivative of $f(x)$; let B_σ be the set of all entire transcendental functions $f(z)$ of exponential type of degree $\leq \sigma$, bounded on the real axis.

By $\bar{\Omega}_n^{(k)}$ we denote the set of all linear operators $V(f, x)$ from $\bar{C}$ into $\bar{C}$ having the property that

$$V(f, x) = f^{(k)}(x),$$

if $f(x)$ is an algebraic polynomial of degree $\leq n$. Along with the set $\bar{\Omega}_n^{(k)}$, consider the set $\bar{\Omega}_{n,n}^{(k)}$ of all linear operators $V_n(f, x)$ from $\bar{C}$ into $\bar{C}$ possessing the properties: 1) for any $f \in \bar{C}$, $V_n(f, x)$ is a polynomial of degree $\leq n$; 2) if $f(x)$ is a polynomial of degree $\leq n$, then $V_n(f, x) = f^{(k)}(x)$. Obviously, $\bar{\Omega}_{n,n}^{(k)} \subset \bar{\Omega}_n^{(k)}$. It is easy to construct such an operator which belongs to $\bar{\Omega}_n^{(k)}$ but does not belong to $\bar{\Omega}_{n,n}^{(k)}$. Therefore $\bar{\Omega}_{n,n}^{(k)} \neq \bar{\Omega}_n^{(k)}$. Put

$$\bar{\rho}_n^{(k)} = \inf_{V\in\bar{\Omega}_n^{(k)}} \|V\|; \qquad \bar{\rho}_{n,n}^{(k)} = \inf_{V_n\in\bar{\Omega}_{n,n}^{(k)}} \|V_n\|.$$

In (1), analogous definitions and notation were given for the periodic case*. As is seen from Theorems 1 and 2, the nonperiodic case differs essentially from the periodic case.

2°. Theorem 1. The equalities

$$\bar{\rho}_n^{(k)} = T_n^{(k)}(1), \qquad k = 0, 1, 2, \dots, n,$$

hold, where $T_n(x) = \cos n \arccos x$. For every $0 \leq k \leq n$ one can specify an operation $V \in \bar{\Omega}_n^{(k)}$ such that $\|\bar{V}\| = \bar{\rho}_n^{(k)}$.

Theorem 2. The equalities

$$\bar{\rho}_{n,n}^{(k)} = T_n^{(k)}(1), \qquad k = 1, 2, \dots, n,$$

hold.

For every $1 \le k \le n$ one can specify an operation $\bar{V}_{n,n}$ such that

$$\|\bar{V}_{n,n}\| = \bar{\rho}^{(k)}_{n,n}.$$

It follows from Theorems 1 and 2 that, for any $k \ge 1$,

$$\bar{\rho}^{(k)}_{n,n}/\bar{\rho}^{(k)}_{n} = 1,$$

whereas in the periodic case

$$\lim_{n\to\infty}\left(\frac{\bar{\rho}^{(k)}_{n,n}}{\bar{\rho}^{(k)}_{n}} : \frac{4}{\pi^2}\ln n\right) = 1.$$

* Only the overbar was absent in the notation.

In the 2π-periodic case, among all linear operators $U_n(f,\theta)$ from C to C that take functions from C into trigonometric polynomials of order n and have the property that $U_n(f,\theta) = f^{(k)}(\theta)$ if $f(\theta)$ is a trigonometric polynomial of order n, the operator $S_n^{(k)}(f,\theta)$ has the smallest norm, where $S_n(f,\theta)$ is the partial sum of order n of the Fourier series of the function $f(\theta)$. Therefore it might seem that in the nonperiodic case, in the class of operators $\overline{\Omega}^{(k)}_{n,n}$, the operator $\bar{\sigma}^{(k)}_n(f,x)$ has the smallest norm, where $\bar{\sigma}_n(f,x)$ is the partial sum of order n of the Fourier series of the function $f(x)$ in the P. L. Chebyshev polynomials $\{T_i(x)\}_{i=0}^{\infty}$. In fact this is not so. A simple calculation shows that the norm of the operator $\bar{\sigma}'_n(f,x)$ satisfies the inequality

$$\|\bar{\sigma}'_n\| \ge cn^2 \ln n. \tag{1}$$

Since the norm of the extremal operator of the class $\overline{\Omega}^{(1)}_{n,n}$ is equal to n^2, it follows from inequality (1) that the operator $\bar{\sigma}'_n(f,x)$ is not extremal in the class $\overline{\Omega}^{(1)}_{n,n}$. From (2) one can obtain that, for any $k \ge 1$, the operator

$$\overline{V}_{n,n}(f,x) = \sum_{j=0}^{n} f(x_j) l_j^{(k)}(x),$$

has the smallest norm in each of the classes $\overline{\Omega}^{(k)}_{n}$ and $\overline{\Omega}^{(k)}_{n,n}$, where $\{l_j(x)\}_{j=0}^{n}$ are the fundamental Lagrange polynomials constructed at the nodes

$$x_j = \cos\frac{j\pi}{n}, \qquad j = 0, 1, 2, \dots, n. \tag{m_0}$$

Thus, for $k \geq 1$ there is an analogy between the operators $S_n^{(k)}(f,\theta)$ and $\overline{V}_{n,n}(f,x)$. It is curious that the problem of computing $\bar{\rho}_{n,n}^{(k)}$ and of finding an extremal operation in the class $\overline{\Omega}_{n,n}^{(k)}$ is closely connected with the problem of finding the best system of nodes for parabolic interpolation. Let $(\mathfrak{M})$ be the set of all possible sequences of numbers

$$-1 \leq x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_0^{(n)} \leq 1. \tag{m}$$

By $\{l_j(x,m)\}_{j=0}^n$ we denote the fundamental Lagrange polynomials corresponding to the points (m). Put

$$M_n^{(k)}(m) = \max_{-1\leq x\leq 1} \sum_{j=0}^{n} |l_j^{(k)}(x)|.$$

From (2) and Theorem 2 the following theorem follows:

Theorem 3. For any $1 \leq k \leq n$, the equalities

$$\inf_{V_n \in \overline{\Omega}_{n,n}^{(k)}} \|V_n\| = \inf_{m\in(\mathfrak{M})} M_n^{(k)}(m) = T_n^{(k)}(1)$$

hold.

For $k = 0$, Theorem 3 ceases to be true, since one can prove that

$$\lim_{n\to\infty} \frac{\lambda_n^{(0)}}{\bar{\rho}_{n,n}^{(0)}} \geq \frac{\pi}{2}, \qquad \text{where } \lambda_n^{(0)} = \inf_{m\in(\mathfrak{M})} M_n^{(0)}(m).$$

3°. Some results from (1) admit an extension to the case of entire transcendental functions of exponential type. Let E be a complete linear normed functional space consisting of functions $f(x)$ defined on the whole real axis $-\infty < x < \infty$. We shall assume that E has the following properties: 1) if $f(x) \in E$, then $f(z)$ is an entire transcendental function of exponential type; 2) if $f(x) \in E$ and y is an arbitrary real number, then $f(x+y) \in E$ and $\|f(x+y)\| = \|f(x)\|$. Such spaces were introduced in (3). It is known (3) that if $f(x) \in E$, then $f^{(k)}(x) \in E$. Let D be the differentiation operator.

By $U(f,x)$ we shall denote an arbitrary linear operator from E into E having the property that, if $f(x) \in B_\sigma$, where $\sigma > 0$ is a fixed number, then

$$U(f,x) = (D\sin\alpha - \sigma\cos\alpha)^k f(x),$$

where α is an arbitrary real number. The set of all such operators will be denoted by $\Omega^{(k)}_{\sigma,\alpha}$. Put

$$\rho^{(k)}_{\sigma,\alpha} = \inf_{U\in\Omega^{(k)}_{\sigma,\alpha}} \|U\|.$$

The question arises of computing $\rho^{(k)}_{\sigma,\alpha}$ and of finding in the class $\Omega^{(k)}_{\sigma,\alpha}$ an extremal operation $\overline{U}$ for which $\|\overline{U}\| = \rho^{(k)}_{\sigma,\alpha}$. The solution of this question is given by the theorem:

Theorem 4. For any real α the equalities

$$\rho^{(k)}_{\sigma,\alpha} = \sigma^k, \qquad k = 0, 1, 2, \ldots$$

hold.

The operation

$$\overline{U}(f,x) = \sigma^k \sum_{j_1,j_2,\ldots,j_k} f\left(x + \sum_{s=1}^{k} \beta_{j_s}\right) \prod_{s=1}^{k} \rho_{j_s}, \tag{2}$$

$$\beta_{j_s} = \frac{j_s\pi - \alpha}{\sigma}, \qquad \rho_{j_s} = (-1)^{j_s-1}\frac{\sin^2\alpha}{(\alpha - j_s\pi)^2}, \qquad -\infty < j_s < \infty, \quad s = 1,2,\ldots,k,$$

belongs to the class $\Omega^{(k)}_{\sigma,\alpha}$ and satisfies the equalities $\|\overline{U}\| = \rho^{(k)}_{\sigma,\alpha} = \sigma^k$.

Corollary. Let $U(f,x)$ be an arbitrary linear operator from E into E having the property that for every $f \in B_\sigma$, $U(f,x) = f^{(k)}(x)$. Then $\|U\| \geq \sigma^k$. Equality is attained for the operation

$$\overline{U}(f,x) = \sigma^k \sum_{j_1,j_2,\ldots,j_k} f\left(x + \sum_{s=1}^{k} \beta_{j_s}\right) \prod_{s=1}^{k} \rho_{j_s},$$

$$\beta_{j_s} = \frac{2j_s - 1}{2\sigma}\pi, \qquad \rho_{j_s} = (-1)^{j_s-1}\frac{4}{(2j_s-1)^2\pi^2}, \qquad -\infty < j_s < \infty,$$

$$s = 1,2,\ldots,k.$$

It is obvious that the corollary follows directly from Theorem 4 when $\alpha = \pi/2$.

Remark. It can be proved that for every $f \in B_\sigma$ the equality

$$(D\sin\alpha - \sigma\cos\alpha)^k f(x) = \overline{U}(f,x),$$

holds, where $\overline{U}(f,x)$ is defined by formula (2). Hence we obtain that for every $f(x) \in B_\sigma$ the inequality

$$\|(D\sin\alpha - \sigma\cos\alpha)^k f\| \leq \sigma^k \|f\| \tag{3}$$

is valid.

For $\alpha = \pi/2$, inequality (3) becomes the well-known inequality of S. N. Bernstein (4). For $k = 1$, inequality (3) is also well known (5).

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4. S. N. Bernstein, Collected Works, vol. 1, Publishing House of the Academy of Sciences of the USSR, 1952, p. 269.

5. G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, part 1, Moscow –Leningrad, 1937, p. 143, problem 165.

Note: Figure translations are in progress. See original paper for figures.