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1961

SovietRxiv

Abstract

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MATHEMATICAL PHYSICS

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SHIFT OF THE BRANCH POINT IN THE MASS OPERATOR OF FORMAL PERTURBATION THEORY FOR A NONIDEAL FERMI-DIRAC SYSTEM

(Presented by Academician N. N. Bogolyubov, 12 VI 1961)

In (1) it was shown that in the formal perturbation-theory series for the energy of the ground state there exist, starting from the 5th order, divergent diagrams, and that in the first high orders the divergences from different diagrams compensate one another. A natural question arises as to whether it makes sense in such a case to study such divergences. Moreover, as shown in (1), irreducible diagrams without self-energy parts never diverge at all, and we can always carry out the so-called mass summation, excluding from consideration all reducible diagrams with self-energy insertions.

The point, however, is that under such a formal rearrangement of the perturbation-theory series there remains a complicated nonlinear integral equation for the new one-particle propagator, which itself is formulated as a series in irreducible diagrams. The study of the divergent reducible diagrams is precisely what gives a way to investigate this equation. In the present note we shall show that the divergences under study are connected with a shift of the branch point in the mass operator, i.e., with an effect which, evidently, cannot be obtained in formal perturbation theory.

First of all, let us note that there exist simple rules for constructing diagrams with denominators for the mass operator, analogous to the rules for constructing vacuum diagrams with denominators for the energy of the ground state. In doing this it is necessary to close the mass diagram for $M(k, E)$ by a fictitious line, which contributes to the denominator $+E$ or $-E$, depending on whether it is a particle or a hole line, in complete analogy with the contribution from a real line. In forming the sign factor, the fictitious line should not be taken into account.

Let us divide the contributions from different diagrams to $M(k, E)$ into two parts: the contributions from diagrams with a particle fictitious line $M_+(k, E)$, and the contributions from diagrams with a hole fictitious line $M_-(k, E)$. The energy denominators in the diagrams for $M_+(k, E)$ and $M_-(k, E)$, according to the remark made above, have the form

$$D_{\pm} = \pm E + \sum \text{ particle energies} - \sum \text{ hole energies}, \tag{1}$$

where the number of particle energies is equal to the number of hole energies, if the particle or hole fictitious line is also counted.

It is easy to see that, according to (1), for the plus case when $E > E(k_F)$, and for the minus case when $E < E(k_F)$, the energy denominators in the diagrams within the region of integration are positive, and, consequently, for each individual diagram of formal perturbation theory

$$\operatorname{Im} M_+(k,E) = 0 \quad \text{for } E > E(k_F); \qquad \operatorname{Im} M_-(k,E) = 0 \quad \text{for } E < E(k_F);$$

this means that in the plus case the diagrams have a cut along the positive part of the E-axis with origin at the branch point $E(k_F)$, and in the minus case ···

diagram has a cut along the negative part of the E axis with its origin at the branch point $E(k_F)$. However, $M_+(k,E)$ and $M_-(k,E)$ have branch points not at all at the unperturbed Fermi energy $E(k_F)$, although each diagram of formal perturbation theory does possess a property of this kind.

The point is that the formal series for $M_+(k,E)$ and $M_-(k,E)$ contain diagrams that diverge at $E = E(k_F)$, whose existence is indicated, in particular, by the divergence criterion formulated in ([1]). According to this criterion these are diagrams with a sufficient number of identical denominators. Thus, diagrams with first-order self-energy insertions will diverge; such diagrams begin to occur from the fifth order on. A partial summation that excludes diagrams with such insertions reduces to replacing $E(k)$ by $E'(k)$ (see (5) from ([1])), which leads to a shift of the branch point in all diagrams from the unperturbed value $E(k_F)$ to the value $E'(k_F)$. In what follows we shall assume that such a summation has been carried out, and we shall omit the prime on $E'(k)$.

Let us consider two possible second-order diagrams for $M_+(k,E)$ and $M_-(k,E)$. We shall make all possible self-energy insertions into these diagrams, as a result of which divergent diagrams will arise:

$$M_+(k,E) = -\frac{1}{V^2}\sum_q \sum_{k'} \frac{v(q)\big(2v(q) - v(k-k'+q')\big)}{-E - E(k') + E(k+q) + E(k'-q) - F_+(k,k';k+q;k'+q;E)}, \tag{2}$$

$$|k+q| > k_F$$
$$|k'| < k_F, \ \ |k'-q| > k_F$$

$$M_-(k,E) = \frac{1}{V^2}\sum_q\sum_{k'}\frac{v(q)(2v(q)-v(k-k'+q))}{E+E(k')-E(k+q)-E(k'-q)-F_-(k,k';k+q;k'-q;E)}, \tag{3}$$

$$|k+q| < k_F$$
$$|k'| > k_F,\ |k'-q| < k_F$$

where F_+ and F_- represent the contributions from parts of diagrams between two identical denominators.

The position of the branch point in (2) and (3) is determined by the minimum and, respectively, the maximum value of E obtained by setting the denominators in (2) and (3) equal to zero at all points inside the integration region. It is reasonable to assume that, in the weak-coupling limit, this maximum or minimum is attained on the boundaries of the integration regions, i.e. on the Fermi surface, since in the absence of interaction this indeed occurs on the Fermi surface. Having thus determined the positions of the branch points E_+ and E_- for $M_+(k,E)$ and $M_-(k,E)$, we shall see that they coincide, and this, together with the inequality $E_+ \geq E_-$, obtained in the case of the ground state of the Lehmann representation, fully justifies the assumption made. On the Fermi surface, from insertions into one line (where such insertions are not representable in the form of two parts connected only by one particle or hole line), we obtain the contributions

$$F'_+ = \sum f_+(E), \qquad F'_- = -\sum f_-(E), \tag{4}$$

where the summation is carried out over possible insertions of the indicated type, or rather over their time versions; $f_+(E)$ and $f_-(E)$ give the contributions from an individual insertion into a particle or hole line. It is not difficult to see that

$$f_+(2E(k_F)-E) = f_-(E). \tag{5}$$

From second-order insertions we obtain

$$\begin{aligned} f_+(E) = &-\frac{1}{V^2}\sum_{q,k'} v(q)(2v(q)\\ &-v(k_F+q-k'))\frac{1}{2E(k_F)-E+E(k')-E(k'-q)-E(k_F+q)}\\ &+\frac{1}{V^2}\sum_{q,k'}\frac{v(q)(2v(q)-v(k_F+q-k'))}{-E-E(k')+E(k'-q)+E(k_F+q)}, \end{aligned} \tag{6}$$

$$|k_F+q|>k_F$$
$$|k'|<k_F,\ |k'-q|>k_F$$

$$\begin{aligned} f_-(E) = &-\frac{1}{V^2}\sum_{q,k'} v(q)(2v(q)-v(k_F+q-k'))\times \\ &\times \frac{1}{E+E(k')-E(k'-q)-E(k_F+q)}+ \\ &+\frac{1}{V^2}\sum_{\substack{q,k' \\ |k_F+q|>k_F \\ |k'|<k_F,\ |k'-q|>k_F}} v(q)(2v(q)-v(k_F+q-k'))\times \\ &\times \frac{1}{E-2E(k_F)-E(k')+E(k'-q)+E(k_F+q)}. \end{aligned} \tag{7}$$

To accuracy up to second order, the position of the branch points E_+ and E_-, according to (2) and (3), is determined from the equations

$$E_+ - E(k_F) + f_+(E_+) + ... = 0, \qquad E_- - E(k_F) + f_-(E_-) + ... = 0, \tag{8}$$

where for $f_+(E)$ and $f_-(E)$ one should use the expressions (6) and (7). From (8) and (5), to the required accuracy, we obtain

$$E_+ = E(k_F) - f_+(E(k_F)) + ..., \qquad E_- = E(k_F) - f_-(E(k_F)) + ... \tag{9}$$

In view of (9) and (5), we see that, to accuracy up to and including the second order, $E_+ = E_-$, i.e., the displacement due to the interaction of the branch points for $M_+(k,E)$ and $M_-(k,E)$ occurs identically to second-order accuracy.

It is easy to see, in fact with the aid of the same arguments based on the use of (5), that to third-order accuracy we shall also have $E_+ = E_-$.

A new situation arises in the fourth order. Therefore we shall dwell on it in somewhat greater detail. Here there are two sources of fourth-order terms in E_+ and E_-. First of all, we must treat the second-order terms more accurately than was done in passing from (8) to (9). Instead of (9) we shall have

$$E_+ = E(k_F) - f_+(E(k_F)) - f_+(E(k_F))f'_+(E(k_F)) + ..., \tag{10}$$

$$E_- = E(k_F) - f_-(E(k_F)) - f_-(E(k_F))f'_-(E(k_F)) + ... \tag{11}$$

It is easy to see that the third-order terms do not give fourth-order terms of a similar type (let us recall that we regard as performed a summation that excludes the proper-energy terms of first order). Next, in the fourth order we must take into account the appearance, in the equations determining the position of the branch points E_+ and E_-, of new terms that are the result of two second-order insertions either into two different lines or into one and the same line. It is sufficient to consider these terms only in the case when both insertions are made in one line, and only for $E = E(k_F)$. It is easy to see that

$$E_+ = E(k_F) - f_2^+ + \dots, \qquad E_- = E(k_F) - f_2^- + \dots, \tag{12}$$

where the notations f_2^+ and f_2^- are explained in (1). Both equations (10) and (11), and equations (12), taken separately, lead to a discrepancy in the motion of E_+ and E_-. Thus, according to (10) and (11), for $\Delta = E_+ - E_-$ we shall have

$$\Delta' = -2f_+(E(k_F))f_+'(E(k_F)) \tag{13}$$

(according to (5), for the derivative with respect to E we have $f_+'(E(k_F)) = -f_-'(E(k_F))$). Also, according to (12) and (6), from (1) we obtain

$$\Delta'' = -f_2^+ + f_2^- = -f_{11}. \tag{14}$$

It is easy to show, however, using (6) and (7), that

$$f_{11} = -2f_+(E(k_F))f_+'(E(k_F)) \tag{15}$$

and that, consequently, the joint consideration of (10), (11), and (12) again leads to $E_+ = E_-$, now already to accuracy up to terms of fourth order.

Although we have not given a general proof that, in all asymptotic orders in the interaction v, the corresponding terms are absent in $\Delta = E_+ - E_-$, the results obtained above indicate this with sufficient persuasiveness. However, this by no means implies that in the asymptotic formula for Δ there are no, say, terms of the form $e^{-1/v}$, which are asymptotically smaller than any positive power of v.

Indeed, everything preceding may be regarded as the beginning of a regular procedure for obtaining an asymptotic formula in integral positive powers of the interaction. Naturally, in such a rough method we cannot speak of terms of the form $e^{-1/v}$. The fact that such terms may exist, at least for an attractive interaction, follows from the theory of superconductivity. It seems to us that, also for repulsive interactions, the question of the existence of such terms still remains open.

Thus, the results obtained testify to the absence of a "coarse" gap proportional to v^n, where n is an arbitrarily large positive integer, but say nothing about the presence or absence of a "thin" gap proportional, for example, to $e^{-1/v}$. In conclusion we note that the fulfillment, for the mass operator $M(k, E)$, of the relation $E_+ = E_-$, and the existence of a compensation of divergent diagrams for ΔE, are connected with one another, which is already sufficiently clear from the arguments presented earlier. Nevertheless, we shall dwell on this question from a somewhat different point of view.

Let us consider the Lehmann representation for the one-particle Green function constructed on the ground state:

$$G(k, E) = \frac{\hbar}{i} \sum_{\alpha} \left\{ \frac{|\varphi_+(k\alpha)|^2}{-E + E_+(k\alpha) - i\varepsilon} + \frac{|\varphi_-(k\alpha)|^2}{-E - E_-(k\alpha) + i\varepsilon} \right\}, \tag{16}$$

$$1 = \sum_{\alpha} \left\{ |\varphi_+(k\alpha)|^2 + |\varphi_-(k\alpha)|^2 \right\}. \tag{17}$$

The previously introduced E_+ and E_- are connected with (16) and (17) in the following way:

$$E_+ = \min_{k\alpha} E_+(k, \alpha), \qquad E_- = \max_{k\alpha} E_-(k, \alpha).$$

It can always be asserted that, for the ground state,

$$E_+ \geq E_-.$$

Using, instead of the zero Green function G_0, the complete Green function G, one can reduce the consideration of all diagrams to the consideration only of irreducible diagrams, without self-energy parts. Using the Lehmann representation for G, one can obtain such irreducible diagrams with energy denominators in which $E_+(k, \alpha)$ and $E_-(k, \alpha)$ will stand. Expanding the energy denominators in the irreducible diagrams in the interaction, we obtain formally divergent diagrams with identical denominators. The expansion has the form

$$\frac{1}{D} = \frac{1}{D_0 + f} = \frac{1}{D_0} - \frac{f}{D_0^2} + \frac{f^2}{D_0^3} + \cdots.$$

Now, if $E_+ = E_-$ (more precisely, such equality is satisfied asymptotically in all orders in integral positive powers of the interaction v^n), then it is easy to see that when D_0 vanishes on the Fermi surface, f also vanishes there (the energy denominator is the difference of equal numbers of particle and hole energies). Consequently, noncompensating divergences cannot arise in ΔE. Suppose $E_+ > E_-$ (more precisely, the difference $E_+ - E_-$ contains asymptotic terms v^n, where n is a positive integer). When D_0 vanishes on the Fermi surface, f does not vanish, and we arrive at noncompensating divergences in ΔE.

The author expresses gratitude to Academician N. N. Bogoliubov for the interest shown in the work, and also to U. I. Safronova for assistance in the work.

Physical-Chemical Institute
named after L. Ya. Karpov

Received
7 VI 1961

CITED LITERATURE

1. V. V. Tolmachev, DAN, **141**, No. 3 (1961).

Note: Figure translations are in progress. See original paper for figures.